Application of the Approximate Inverse to 3D X–Ray CT and Ultrasound Tomography

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1 Introduction

We study operator equations $Af = g$ for operators between Hilbert spaces X and Y. Both the cases of linear operators A and of nonlinear A with a special structure are treated. Approximate inverse means a solution operator which maps the data g to a stable approximation of the solution of the ill – posed problem $Af = g$. This inversion operator is precomputed without using the data g, see [9].

The method is based on two ideas. First, the computation of moments of the solution is stable; i.e., we compute instead of f the approximation $\langle f, e_\gamma \rangle$ with a suitable mollifier e_γ reducing in that way the high frequency components in the solution which are mostly affected by the data noise. This can be reformulated as using a weaker topology in the space X, see [3], [6]. Examples for e_γ are given in the next section, e_γ can be a basis function for projection methods, it can be chosen such that $\langle f, e_\gamma \rangle$ approximates a derivative of f; in wavelet language it can be a scaling function or a wavelet. Second, in the case of linear operators the computation of $\langle e_\gamma, f \rangle$ is then achieved by approximating e_γ in the range of the adjoint operator A^* by the reconstruction kernel $\psi_\gamma : A^*\psi_\gamma \simeq e_\gamma$. Then

$$\langle f, e_\gamma \rangle \simeq \langle f, A^*\psi_\gamma \rangle = \langle Af, \psi_\gamma \rangle = \langle g, \psi_\gamma \rangle \ .$$

Definition 1. The operator $S_\gamma : Y \to X$ defined as

$$S_\gamma g(x) = \langle g, \psi_\gamma(x) \rangle \tag{1}$$

where ψ_γ solves for a given and sufficiently smooth mollifier e_γ either

$$A^*\psi_\gamma(x) = e_\gamma(x, \cdot) \quad \text{for} \quad e_\gamma(x, \cdot) \in \mathcal{R}(A^*) \tag{2}$$

or

$$AA^*\psi_\gamma(x) = Ae_\gamma(x, \cdot) \ , \tag{3}$$

where A acts on the second variable of e_γ, is called *approximate inverse* and $\psi_\gamma(x)$ is called the *reconstruction kernel* for the point x.

This is the mollifier method presented in [10]. It also has been used to accelerate convergence for finite element solutions in [5]. The recently introduced method in smoothed particle hydrodynamics is based on the same ideas.
We show that the approximate inverse is a regularization method. For nonlinear operators we combine these ideas with the results of Snieder [18] who generalized the Backus – Gilbert method to nonlinear problems. A detailed analysis of the method shows that the computational effort is much smaller than in [18]. Applications to 3D x–ray CT and ultrasound CT conclude the paper.

2 Approximative Inverse for Linear Problems

In the following we assume A to be a linear, continuous operator between the Hilbert spaces X and Y. Especially we think of X as a space of functions and Y as a finite dimensional space of measurements. Hence, if necessary, we use $X = L_2(\Omega)$ for a suitable set $\Omega \subset \mathbb{R}^d$. Examples for mollifiers are

$$e_\gamma(x, y) = \frac{d}{\mathrm{vol}(S^{d-1})\gamma^d}\chi_\gamma(x - y) \tag{4}$$

where χ_γ is the characteristic function of the ball around 0 with radius γ and $\mathrm{vol}(S^{d-1})$ is the measure of the surface of the unit ball in $\mathbb{R}^d$. Here local averages of the solution are computed. With the band limiting filter

$$e_\gamma(x, y) = \left(\frac{\gamma}{\pi}\right)^d \mathrm{sinc}(\gamma(x - y)) \tag{5}$$

the high – frequency components in the solution are eliminated. Fast decaying is the kernel of the heat equation

$$e_\gamma(x, y) = (2\pi)^{-d/2}\gamma^{-d}\exp(-|x - y|^2/(2\gamma^2)) \ . \tag{6}$$

In all cases the parameter γ acts as a regularization parameter. The mollifier e_γ is not necessarily a function with mean value 1. When the essential information we need are discontinuities in f we can use as e_γ a function such that $\langle f, e_\gamma(x, \cdot)\rangle$ approximates a derivative of $f(x)$. This means that e_γ can also be a wavelet, see e.g. [12].
First we assume the equation $A^*\psi_\gamma = e_\gamma$ to be solvable. Then we put

$$\langle f, e_\gamma\rangle = \langle f, A^*\psi_\gamma\rangle = \langle Af, \psi_\gamma\rangle = \langle g, \psi_\gamma\rangle =: S_\gamma g \ . \tag{7}$$

This is the technique to derive inversion formulas in x – ray computer tomography resulting in the so – called filtered backprojection methods, see e.g. [8], [14]. If the equation $A^*\psi_\gamma = e_\gamma$ is not solvable we approximate ψ_γ by minimizing the defect $\|A^*\psi_\gamma - e_\gamma\|$ for sufficiently smooth e_γ leading to the equation

$$AA^*\psi_\gamma = Ae_\gamma \ . \tag{8}$$

Then we get

$$\langle f, e_\gamma\rangle \simeq \langle f, A^*\psi_\gamma\rangle = \langle Af, \psi_\gamma\rangle = \langle g, \psi_\gamma\rangle =: S_\gamma g \ .$$

It is important to mention that no artificial discretization of f is needed as introduced by projection methods, see e.g. [14], [15], [17]. For the numerical computation of ψ_γ the matrix AA^* needs a coarse stabilization, the fine tuning is then achieved by the choice of γ, compare [16].

3 Comparison with other Methods

Now let A be a compact operator between the Hilbert spaces X and Y. Then it has a singular value decomposition

$$\{v_n, u_n; \sigma_n\}_n$$

where v_n, u_n are normalized and

$$Av_n = \sigma_n u_n \quad \text{and} \quad A^* u_n = \sigma_n v_n \ .$$

Regularization methods, like Tikhonov – Phillips, truncated singular value decomposition or Landweber iteration, have the form

$$T_\gamma g = \sum_n F_\gamma(\sigma_n)\sigma_n^{-1}\langle g, u_n\rangle v_n \ , \tag{9}$$

see for example[2], [4], [6]. The following result shows that these methods are special cases of the approximate inverse.

Theorem 2. *Let the regularization method T_γ in (9) be given with a filter F_γ. Then this method can be written as an approximate inverse with mollifier*

$$e_\gamma(x, y) = \sum_n F_\gamma(\sigma_n) v_n(x) v_n(y) \ . \tag{10}$$

Proof: The definition of ψ_γ as solution of $AA^*\psi_\gamma = Ae_\gamma$ in (3) leads with

$$\psi_\gamma(x) = \sum_n \sigma_n^{-1}\langle e_\gamma(x, \cdot), u_n\rangle v_n$$

to

$$\psi_\gamma(x) = \sum_n F_\gamma(\sigma_n)\sigma_n^{-1} u_n v_n(x) \ .$$

Then

$$S_\gamma g(x) = \langle g, \psi_\gamma(x)\rangle = T_\gamma g(x) \ .$$

In contrast to the Backus – Gilbert method, see [1], [17], the matrix for computing the reconstruction kernel does not depend on the reconstruction point. Also there is a the possibility to pattern the reconstruction kernel in almost any desirable way, one is not forced to approximate the delta distribution with a kernel like $|x - y|^{-2}$.

4 Regularization Properties of the Approximate Inverse

In the following we show that the approximate inverse is a regularization method. This means the following.

Definition 3. A *regularization* of $A^\dagger$ is a family of operators

$$\{R_\gamma\}_{\gamma>0}\ ,\ \ R_\gamma : Y \to X$$

with the following properties:
there exists a mapping $\gamma : \mathbb{R}_+ \times Y \to \mathbb{R}_+$ such that for all $g \in \mathcal{R}(A)$ and for all $g^\varepsilon \in Y$ with $\|g^\varepsilon - g\| \leq \varepsilon$

$$\lim_{\varepsilon\to 0} R_{\gamma(\varepsilon, g^\varepsilon)} g^\varepsilon = A^\dagger g\ . \tag{11}$$

There are different methods to compare regularizations. In order to do this we need some notion. We define the Hilbert spaces

$$X_\nu = \mathcal{R}(A^* A)^{-\nu/2}$$

equipped with the norm

$$\|f\|_\nu = \Big(\sum_n \sigma_n^{-2\nu} |\langle f, v_n\rangle|^2\Big)^{1/2}$$

for real ν. In a similar way we define norms on Y. For $\nu = 0$ we simply write X and $\|\cdot\|$. Note that

$$\|Af\|_\nu = \|f\|_{\nu-1}\ \text{and} \|A^\dagger g\|_\nu = \|g\|_{\nu+1}\ .$$

We use the following definition.

Definition 4. A regularization method is called *order-optimal* of order ν if there exists a constant c such that

$$\|R_\gamma g^\varepsilon - A^\dagger g\| \leq c\varepsilon^{\nu/(\nu+1)}\rho^{1/(\nu+1)} \tag{12}$$

for all $g \in Y_\nu$ with $\|A^\dagger g\|_\nu \leq \rho$.

Now we show that the approximate inverse is a regularization in the above mentioned sense. In order to avoid technical difficulties we assume in the following that for each x the mollifier $e_\gamma(x,\cdot)$ is in the range of A^*. Then we can represent

$$e_\gamma(x,\cdot) = \sum_n e_{\gamma,n}(x) v_n$$

with

$$e_{\gamma,n}(x) = \langle e_\gamma(x,\cdot), v_n\rangle\ .$$

From this we get with (2) the reconstruction kernel

$$\psi_\gamma(x) = \sum_n \sigma_n^{-1} e_{\gamma,n}(x) u_n$$

resulting in

$$S_\gamma g(x) = \sum_n \sigma_n^{-1} \langle g, u_n \rangle e_{\gamma,n}(x) \ .$$

Hence we decompose the operator S_γ into $E_\gamma A^\dagger$ where $A^\dagger$ is the generalized inverse given as

$$A^\dagger g = \sum_n \sigma_n^{-1} \langle g, u_n \rangle v_n$$

and

$$E_\gamma : \mathcal{N}(A)^\perp \to X$$

with

$$E_\gamma f(x) = \sum_n \langle f, v_n \rangle e_{\gamma,n}(x)$$

Theorem 5. *Let $E_\gamma : X \to X_{-1}$ be continuous for each γ; i.e., there exists a constant $c(\gamma) > 0$ such that*

$$\|E_\gamma f\| \le c(\gamma) \|f\|_{-1} \quad \textit{for all} \quad f \in \mathcal{N}(A)^\perp \tag{13}$$

and

$$\lim_{\gamma \to 0} \|E_\gamma f - f\| = 0 \tag{14}$$

for all $f \in \mathcal{N}(A)^\perp$. Then $S_\gamma = E_\gamma A^\dagger$ is a regularization of $A^\dagger$.

Proof. Let $g \in \mathcal{R}(A)$ and $g^\varepsilon \in Y$ with $\|g^\varepsilon - g\| \le \varepsilon$, then

$$\begin{aligned} \|S_\gamma g^\varepsilon - A^\dagger g\| &\le \|S_\gamma (g^\varepsilon - g)\| + \|(S_\gamma - A^\dagger) g\| \\ &= \|E_\gamma A^\dagger (g^\varepsilon - g)\| + \|(E_\gamma - I) A^\dagger g\| \\ &\le c(\gamma) \|A^\dagger (g^\varepsilon - g)\|_{-1} + \|(E_\gamma - I) A^\dagger g\| \\ &\le c(\gamma) \varepsilon + \|(E_\gamma - I) A^\dagger g\| \end{aligned}$$

which completes the proof.

In a similar manner we can show the order–optimality of the approximate inverse.

Theorem 6. *Let $E_\gamma : X \to X_{-1}$ be continuous with*

$$\|E_\gamma f\| \le c \gamma^{-\alpha} \|f\|_{-1} \quad \textit{for all} \quad f \in \mathcal{N}(A)^\perp \tag{15}$$

and

$$\|E_\gamma f - f\| \le c_{\nu^*} \gamma^{\alpha \nu^*} \|f\|_{\nu^*} \tag{16}$$

for all $f \in X_{\nu^*}$ and a $\alpha \in \mathbb{R}$. Then the method is order–optimal for all $0 \le \nu \le \nu^*$.

The proof follows the ideas of the above theorem and techniques given for example in [6].

For the classical regularization methods we have

$$E_\gamma v_n = F(\sigma_n) v_n \tag{17}$$

and then the order–optimality follows from conditions for the filter F_γ given for example in [6]. Again we see that this is a special case of the approximate inverse where

$$E_\gamma v_n = \langle e_\gamma(x, \cdot), v_n \rangle \ .$$

For the approximate inverse we can use results for mollifiers to prove order–optimality.

5 Efficient Implementation

If the problem shares some invariance properties they can be used for a fast realisation of the method. Let in the following E_γ be a function of the variable y only. We can think of $E_\gamma(y) = e_\gamma(0, y)$; i.e., a mollifier concentrated around 0 which then may be shifted to arbitrary points as $e_\gamma(x, y) = E_\gamma(x - y)$.

Theorem 7. *Let $A : X \to Y$ and let T_1^x be a linear operator on X and T_2^x, T_3^x be linear operators on Y such that*

$$A T_1^x = T_2^x A \tag{18}$$

and

$$T_2^x A A^* = A A^* T_3^x \ . \tag{19}$$

Let w_γ be the minimum norm solution of

$$A A^* w_\gamma = A E_\gamma \ . \tag{20}$$

Then the minimum norm solution of

$$A A^* \psi_\gamma(x) = A T_1^x E_\gamma$$

is

$$\psi_\gamma(x) = T_3^x w_\gamma \ . \tag{21}$$

Proof: From the invariance properties follows

$$\begin{aligned} A T_1^x E_\gamma &= T_2^x A E_\gamma \\ &= A A^* T_3^x w_\gamma \end{aligned}$$

which completes the proof.

This means that only the solution w_γ has to be computed and stored, the kernels for other reconstruction points x are found by the action of T_1^x on E_γ and by

T_3^x on w_γ.

In the case of a finite number of data where $(Af)_n = Af(x_n)$, $n = 1, \ldots, N$ for suitable points x_n the reconstruction kernel w_γ is a vector in $\mathbb{R}^N$ with $(w_\gamma)_n = w_\gamma(x_n)$. Then $w_\gamma(x_n - x)$ can be evaluated by linear interpolation between $(w_\gamma)_m$ and $(w_\gamma)_{m+1}$ with $x_m \leq x_n - x < x_{m+1}$.

For $Y = \mathbb{R}^N$ and M reconstruction points the storage needs is $M \times N$ real numbers. If the problem has invariance properties this can be dramatically reduced. If translation invariance for example holds only N real numbers have to be stored!

6 Nonlinear Problems

We start the presentation of the approach for the nonlinear problem

$$A : L_2(\Omega) \to \mathbb{R}^N$$

by considering for the sake of simplicity only a quadratic problem. Let the operator A be given as

$$Af = A_1 f + A_2 f$$

where A_1 is linear,

$$(A_1 f)_n = \int_\Omega k_n^1(y) f(y) dy$$

and A_2 is a quadratic operator defined as

$$(A_2 f)_n = \int_\Omega \int_\Omega k_n^2(y_1, y_2) f(y_1) f(y_2) dy_1 dy_2 \ .$$

With k^1 we denote the vector of the N components of k_n^1 and similarly with k^2 the vector of the N components of k_n^2. For the approximate inverse we follow Snieder [18] and make the following ansatz

$$f_\gamma(x) = \langle g, \psi_\gamma(x) \rangle + \langle g, V_\gamma(x) g \rangle$$

where $V_\gamma(x)$ is an $N \times N$ matrix. We replace g by Af and get

$$f_\gamma(x) \simeq \langle A_1 f, \psi_\gamma(x) \rangle + \langle A_2 f, \psi_\gamma(x) \rangle + \langle A_1 f, V_\gamma(x) A_1 f \rangle \qquad (22)$$

where we omitted the higher order terms.

In the following we always use $x \in \Omega$ as the reconstruction point which is arbitrarily fixed; and with y_1, y_2 and so on we denote the integration variables in the integral operators.

Of course we can attempt to approximate with the right – hand side an expression like $\langle f, e_\gamma(x, \cdot) \rangle$. But then the approximate inverse is not independent of the data, this does not lead to an approximate inverse as aimed for. We therefore follow

[18] and consider the terms separately. We approximate the mollified solution with the linear term as well as possible which means

$$\langle A_1 f, \psi_\gamma(x)\rangle \simeq \langle f, e_\gamma(x,\cdot)\rangle \ ,$$

hence

$$A_1 A_1^* \psi_\gamma = A_1 e_\gamma \ .$$

The computation of ψ_γ thus follows exactly the lines as in the linear case. When this term presents the solution it only remains to make the rest as small as possible. Denoting the remainder by

$$R(x)(y_1,y_2) = \psi_\gamma(x)^\top k^2(y_1,y_2) + k^1(y_1)^\top V_\gamma(x) k^1(y_2) \tag{23}$$

where $a^\top b$ is the scalar product in the image space $\mathbb{R}^N$, we can write the rest in (22) as

$$\int_\Omega \int_\Omega R(x)(y_1,y_2) f(y_1) f(y_2) dy_1 dy_2 \ .$$

In order to make this term as small as possible we minimize with respect to $V_\gamma(x)$ for each reconstruction point x the norm

$$||R(x)(\cdot,\cdot)||^2_{L_2(\Omega\times\Omega)} \to \min \ .$$

We get

$$\begin{aligned}
||R(x)||^2 &= \int_{\Omega\times\Omega} \Big(\psi_\gamma^\top(x) k^2(y_1,y_2) + k^1(y_1)^\top V_\gamma(x) k^1(y_2)\Big)^2 dy_1 dy_2 \\
&= \int_{\Omega\times\Omega} \Big((\psi_\gamma(x)^\top k^2(y_1,y_2))^2 \\
&\quad +2\psi_\gamma(x)^\top k^2(y_1,y_2) k^1(y_1)^\top V_\gamma(x) k^1(y_2) \\
&\quad +(k^1(y_1)^\top V_\gamma(x) k^1(y_2))^2\Big) dy_1 dy_2 \ .
\end{aligned}$$

Differentiating this expression with respect to the fixed matrix element $(V_\gamma)_{ij}(x)$ and equating the derivative to zero we find

$$\begin{aligned}
&\int_\Omega k_i^1(y_1) k^1(y_1)^\top dy_1 \ V_\gamma(x) \ \int_\Omega k_j^1(y_2) k^1(y_2)^\top dy_2 \\
&= -\int_{\Omega\times\Omega} \psi_\gamma^\top(x) k^2(y_1,y_2) k_i^1(y_1) k_j^1(y_2) dy_1 dy_2 \ , i,j = 1,\ldots,N \ .
\end{aligned}$$

In matrix notion this is simply

$$(A_1 A_1^*) V_\gamma(x) (A_1 A_1^*) = -\sum_{n=1}^{N} \psi_{\gamma,n}(x) B_n \ , \tag{24}$$

where the matrices B_n on the right – hand side are

$$B_n = \int_{\Omega\times\Omega} k^1(y_1) k_n^2(y_1,y_2) k^1(y_2)^\top dy_1 dy_2 \ .$$

This means that with $C_n = (A_1A_1^*)^{-1}B_n(A_1A_1^*)^{-1}$ the function $k^{1^\top}(y_1)C_nk^1(y_2)$ is the orthogonal projection of the function $k_n^2(y_1,y_2)$ on the linear space

$$\mathrm{span}\{k_i^1(y_1)k_j^1(y_2) : i,j = 1,\ldots,N\} .$$

Hence, if the k_n^1 are linearly independent, the matrix V_γ is simply

$$V_\gamma(x) = -\sum_{n=1}^{N} \psi_{\gamma,n}(x)C_n$$

and we get

$$f_\gamma(x) = \langle g, \psi_\gamma(x)\rangle - \sum_{n=1}^{N} \psi_{\gamma,n}(x)\langle g, C_n g\rangle .$$

It is important for the implementation that the matrices C_n are independent of the reconstruction points x. Hence the complexity is determined by the linear part. An example for identifying a vibration string from relative shifts in the eigenvalues is reported in [9].

7 3D X-Ray CT

In [11] we presented a method to reconstruct contours from 3D x–ray data when the x–ray source is moved on a circle around the object. It is fired at 400 positions and the cone of rays is detected at an array of 512×512 detectors resulting in more than 100 million data.

As mollifier we chosed a wavelet such that

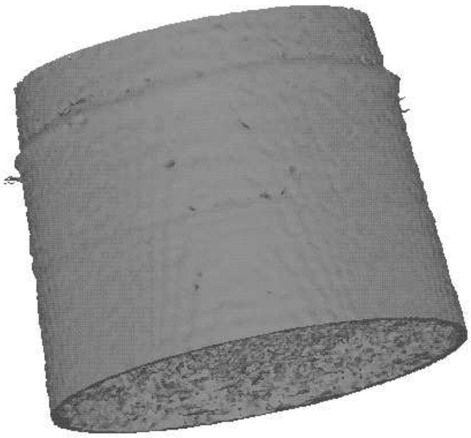
$$\hat{e}(\xi) = ||\xi||h(\xi)$$

for a suitable h resulting in a formula of filered backprojection type

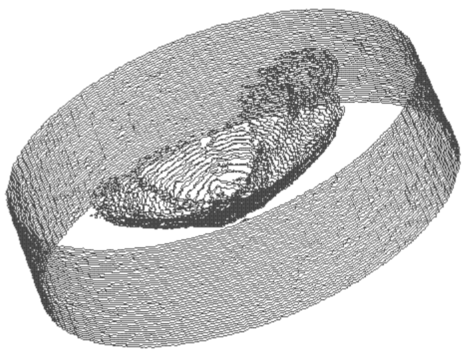
$$f_\gamma(x) \simeq \Delta_x \int_\Gamma ||x-a||^{-2} Df\Big(a, \frac{x-a}{||x-a||}\Big)da \tag{25}$$

where Γ denotes the curve where the x–ray source is moved and $Df(a,\theta)$ is the data from source position a and direction $\theta \in S^2$.

a



b



Reconstruction from 10^8 x–ray data, measured at the Fraunhofer Institut für zerstörungsfreie Prüfverfahren, Saarbrücken
Fig. a: Ceramical valve, Fig. b: Display of the interior inclusion of air caused by the production, invisible from the outside.

8 Ultrasound Tomography

We present a technique to apply the approximate inverse to an inverse scattering problem for the Helmholtz equation. In the following we first consider the case of just one incident plane wave in direction Θ. We define the mapping

$$A : L_2(\Omega) \to L_2(\Gamma)$$

where we first put $\Omega = \Gamma = \mathbb{R}^3$, that means we make no use of the fact that f is compactly supported. Let G be the Greens function for the Helmholtz equation, then

$$A\Phi(\eta) = \int_\Omega G(|\eta - y|)\Phi(y)dy \tag{26}$$

which is a convolution equation. From the last section we know that also the reconstruction is of displacement type using $T_1 = T_2 = T_3 = D^z$ where $D^z f(x) = f(x - z)$ in Theorem 7 if $e_\gamma(x, y) = D^x E_\gamma(y) = E_\gamma(y - x)$. The mapping AA^* is generated by the kernel

$$G_2(\eta - \xi) = \int_{\mathbb{R}^3} G(|\eta - x|)G(|\xi - x|)dx \ .$$

Defining with the unitary matrix U the operator D^U as $D^U f(x) = f(Ux)$ we compute that $AD^U = D^U A$ and $AA^* D^U = D^U AA^*$, leading to another invariance and a simplification of the reconstruction kernel ψ_γ for circular symmetric $E_\gamma(|y|)$.

After solving $AA^*\psi_\gamma(x, \cdot) = Ae_\gamma(x, \cdot)$ for the reconstruction point x we put for the data given on Γ

$$\Phi_\gamma(x) = \langle u^s, \psi_\gamma(x, \cdot)\rangle_{L_2(\Gamma)} = \int_\Gamma u^s(\eta)\psi_\gamma(|x - \eta|)d\eta \ . \tag{27}$$

Evaluating the integral in (26) at the point $x \in \Omega$ we find an approximation for the scattered field inside the scatterer as

$$\begin{aligned} u^s_\gamma(x) &:= \int_{\mathbb{R}^3} G(|x - y|)\Phi_\gamma(y)dy \\ &= \int_\Gamma u^s(\eta) \int_{\mathbb{R}^3} G(|x - y|)\psi_\gamma(|y - \eta|)dyd\eta \\ &= \int_\Gamma K_\gamma(x - \eta)u^s(\eta)d\eta \ . \end{aligned}$$

We observe that also the kernel K_γ can be precomputed independant of the data. Then we put

$$\begin{aligned} f_\gamma(x) &:= \Phi_\gamma(x)/(u^i(x) + u^s(x)) \\ &= \frac{\int_\Gamma u^s(\eta)\psi_\gamma(|x - \eta|)d\eta}{e^{i\Theta x} + \int_\Gamma u^s(\eta)K_\gamma(x - \eta)d\eta} \ . \end{aligned}$$

For multiple incoming plane waves we average over these values, resulting in a discretization of

$$f_\gamma(x) = \frac{1}{4\pi} \int_{S^2} \frac{\int_\Gamma u^s(\Theta, \eta)\psi_\gamma(|x - \eta|)d\eta}{e^{i\Theta x} + \int_\Gamma u^s(\Theta, \eta)K_\gamma(x - \eta)d\eta} d\Theta \ . \tag{28}$$

The solution of the ill-posed linear problem is achieved by precomputing the reconstruction kernel ψ_γ and based on this also the kernel K_γ. Hence the reconstruction is realized by a fast implementation of this formula of filtered backprojection type used in x-ray tomography, compare [8], [14].

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