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Uncertainty, Ghosts and Resolution in Radon Problems

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Abstract: We study the nonuniqueness problem for Radon transforms for finitely many directions. In the early days of the application of computed tomography they caused some confusion about the possible information content in the reconstructions from tomographic data. The existence of nontrivial functions in the null space started the analysis of these then so-called ghosts. A result of Logan [25] described properties of the spectrum of those functions. Only with the description of those functions in terms of special functions by Louis [27] a more detailed study and an improvement of earlier results was possible. Here we describe the essential steps to find those characterizations and the analysis of the spectral properties allowing for resolution results.

Keywords: Radon transform, nonuniqueness for finitely many directions, resolution, ghosts

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1 Introduction

After the pioneering work of Hounsfield and Cormack introducing computed tomography the practical consequences of the limitations of the measuring system started moving in the focus of interest. The mathematical model of X-ray computed tomography in two dimensions is given by the two-dimensional Radon transform

$$\mathbf{R}f(\omega,s) = \int_{\mathbb{R}^2} f(x)\delta(s - x^\top \omega)dx$$

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where f is the searched-for density distribution, ω is the direction orthogonal to the propagation of the x-ray beams, and s is the signed distance of the ray from the origin. If $\mathbf{R}f(\omega,\cdot)$ is given for a direction ω and all $s \in \mathbb{R}$ then those data are called complete projections. A first example of a non-trivial function whose complete projections are 0 was constructed on a square where on the four equally sized subquarters the function has the values $+1$ or -1 alternating and the two measuring directions are parallel to the sides of the squares. This shows that the Radon transform for those directions has a non-trivial null space. The argument that typical densities are positive does not prevent from nonuniqueness, because when one adds such a null space function to another density such that the sum is positive, there is still nonuniqueness as the data are the same.

Those considerations lead Katz [21] to study the nonuniqueness problem in the space of piecewise constant functions on a regular pixel basis as used in those days for ART, the algebraic reconstruction technique, see e.g. Herman [19].

Another approach was followed in the group of K.T. Smith [48] in Corvallis. They described the functions in the null space as the solution of a system of homogeneous hyperbolic equations. The result he reformulated as: *A finite set of radiographs tells nothing at all*. Here radiographs stands for complete projections. With that remark Smith tried to stop several projects in the 70ies, but fortunately enough his result was shown too late, already one decade of positive experience with X-ray tomography existed.

In a next step characterizations of the functions in the null space were addressed. In the meantime the wording ghosts for those functions came in use. Indeed this is in contrast to the everyday life: there some people see ghosts but everyone knows that they do not exist. Here one can prove that they exist but they are invisible for certain directions.

The first remarkable characterization result was shown by Logan [25] in a paper with some 500 formulas. Logan studied the frequency distribution of the ghosts and showed that for the two-dimensional case the largest part of the spectrum is contained outside of a ball around 0 with radius equal to the number of complete projections. Based on consistency conditions for the Radon transform it then was possible to give a constructive description of the functions in the null space and hence to derive the results of Logan in a more elegant form and to improve the estimates, see Louis [27].

Some related result should be mentioned here. The Radon transform for the exterior problem in $\mathbb{R}^N$ with suitable weight, where planes do not intersect the interior of the unit ball was studies by Quinto [40]. For the X-ray transform a singular value decomposition and the case of a finite number of data with resolution properties was a groundbreaking work by Maass, [35]. Null space results for cone-beam tomography were given by Derevtsov, [11]. The singular value decompsitions for the cone beam transform, see Kazantsev, [22] and Quellmalz, Hielscher and Louis, [39] have to be exploited in this direction, yet. For line integrals on Cormack-type of curves the nonuniqueness

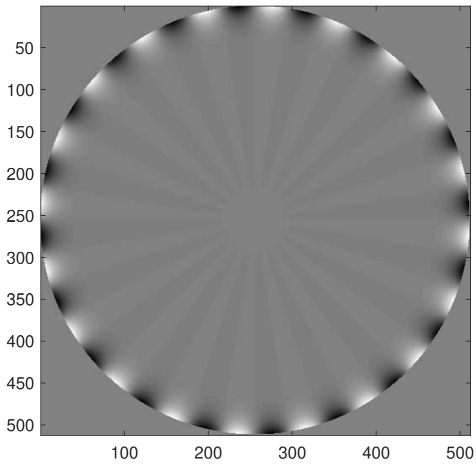

Fig. 1: Function in the null space with smallest index for 20 equidistributed directions in a ball around 0 with radius 1.

problem was considered by Rigaud, [41]. In case of incomplete data problems even for an infinite number of projections there is nonuniquess for the case of truncated projection, where only those line integrals are given which intersect a ball $V(0,a) \subset V(0,1)$ with $a < 1$, as shown by Smith, Solmon and Wagner, [48]. Functions in the corresponding null space were constructed by Louis and Rieder, [33]. Recently Hahn, [16], studied dynamic tomographic problems where the structure of the object moves during the measurements. With a far reaching extension of the methods presented here, she characterized the achievable resolution. For the case of vector tomography the situation is more complicated, there exist nonuniqueness problems even for an infinite number of data, see Sharafutdinov, [45] or Schuster, [42]. Also the singular value decomposition for tensor tomography, presented by Derevtsov, Efimov, Louis and Schuster, [12], has yet to be exploited. In the case of discrete tomography ghosts were used to improve the resolution, see e.g. Chandra, Svalbe et. al. [7]. The list of contributions to this field cannot be complete, due to magnificent results achieved in the last 40 years. As recent contributions we mention [17, 23, 1, 5].

In this paper we present in the next section the problem of nonuniqueness of the Radon transform for finitely many directions. We derive the main ideas for the two-dimensional Radon transform, where we derive series expansions for the functions in the null space, the co-called ghosts. We then extend the results to arbitrary dimension and elaborate on the main differences. The third section is devote to the question of resolution, directly for arbitrary dimensions where the spectral properties of the ghosts are determined, allowing for analyzing the resolution. Finally an outlook on increasing the resolution by introducing additional information including compressive sensing and deep learning.

2 The Ghosts

2.1 Properties of the Null Space

In the following we concentrate on the two-dimensional case, peculiarities of the higher dimensional case are mentioned at the end of this section. We consider the Radon transform as continuous mapping between L_2 spaces, say

$$\mathbf{R} : L_2(\Omega_2) \mapsto L_2(C_2) \tag{1}$$

$$\mathbf{R} : L_2(\Omega_2) \mapsto L_2(C_2, w_1^{-1}) \tag{2}$$

where Ω_2 denotes the unit ball in $\mathbb{R}^2$, C_2 is the unit cylinder

$$C_2 = S^1 \times [-1,1]$$

with S^1 the unit sphere in $\mathbb{R}^2$ and w_1 is the weight

$$w_1(s) = (1-s^2)^{1/2} .$$

The continuity in (2) follows by straightforward calculation, the other simply by the continuous embedding of the space with weight in the one without weight.
We now consider the null space for finitely many directions. Let

$$\mathscr{A}_p = \{\omega_1, \dots \omega_p\} \subset S^1 \tag{3}$$

be a set of p distinct directions. The distribution of the directions on the unit circle plays no role for the uniqueness questions, whereas for the stability of the inversion it is crucial.
Next we consider the null space of the Radon transform for those directions

$$\mathscr{N}_p = \mathscr{N}(\mathscr{A}_p) = \{f \in L_2(\Omega_2) : \mathbf{R}f(\omega,s) = 0 \text{ for all } \omega \in \mathscr{A}_p \text{ and almost all } s \in [-1,1]\} \tag{4}$$

We note some obvious informations.

Lemma 2.1. *Let f be in the null space $\mathscr{N}(\mathscr{A}_p)$. Then*
i) if T^a is a translation such that $T^a f(x) = f(x-a)$, then $T^a f \in \mathscr{N}(\mathscr{A}_p)$.

ii) if $t \neq 0$ and D^t is a dilation such that $D^t f(x) = f(x/t)$ then $D^t f \in \mathscr{N}(\mathscr{A}_p)$.
iii) if D^U is a rotation such that $D^U(\mathscr{A}_p) \subset \mathscr{A}_p$ and $D^U f(x) = f(Ux)$, then $D^U f \in \mathscr{N}(\mathscr{A}_p)$.

Proof. The proof follows from properties of the Radon transform. For i) we observe that $\mathbf{R}T^a f(\omega,s) = \mathbf{R}f(\omega, s-a^\top\omega)$. Hence, if $\mathbf{R}f(\omega,\cdot) = 0$, then also $\mathbf{R}f(\omega,s-a^\top\omega) = 0$. For the dilation in ii) we use $\mathbf{R}D^t(\omega,s) = \frac{1}{t}\mathbf{R}f(\omega,\frac{s}{t}) = 0$ with the same argument as above. Finally for the rotation in iii) we get $\mathbf{R}D^U f(\omega,s) = \mathbf{R}f(D^U\omega,s) = 0$. □

The following presentation follows Louis [27]. Occasionally this paper is referenced as authered by Louis-Törnig. But the esteemed colleague Prof. Törnig, who accepted the paper as member of the editorial board of that journal for publication is correctly mentioned as 'communicated by W. Törnig' on the printed version. Only much later he was made by some ignoramus, probably a computer, to a co-author in some lists. The main ingredient for finding a representation of functions in the null space is now the consistency conditions named after Gel'fand-Graev-Vilenkin [14] resp. Helgason [18] or Ludwig [34] depending on the background of the authors, resp. It states, among others, that when p_m is a polynomial of degree m in s then

$$\int \mathbf{R}f(\omega,s)p_m(s)ds = q_m(\omega)$$

with a polynomial q_m of degree m in the directions. With the weight in the mapping property (2) the use of Chebyshev polynomials of the second kind, U_m, is promising. These polynomials form a complete orthogonal system on $L_2([-1,1],w_1)$. In order to apply the consistency conditions we consider

$$p_m = w_1 U_m$$

and the weighted L_2 space from (2) to find

$$\langle \mathbf{R}f(\omega,\cdot), w_1 U_m\rangle_{L_2([-1,1],w_1^{-1})} = \int_{-1}^{1} \mathbf{R}f(\omega,s)U_m(s)ds = q_m(\omega)\ . \tag{5}$$

With these considerations we can decompose the functions in the range of the Radon transform as

$$\mathbf{R}f(\omega,s) = w_1(s)\sum_{m=0}^{\infty} U_m(s)q_m(\omega) \tag{6}$$

where the expansion coefficients are, taking the orthogonality of the Chebyshev polynomials into account,

$$q_m(\omega) = \frac{2}{\pi}\int_{-1}^{1} \mathbf{R}f(\omega,s)U_m(s)ds \tag{7}$$

$$= \sum_{\substack{\lambda=-m \\ m+\lambda \text{ even}}}^{m} c_\lambda^m e^{\imath\lambda\varphi} \tag{8}$$

with coefficients $c_\lambda^m \in \mathbb{C}$. where we additionally used that the consistency conditions also state that $\mathbf{R}f(-\omega,-s) = \mathbf{R}f(\omega,s)$, hence the q_m are even (odd) when m is even(odd) as the Chebyshev polynomials.

In the following we use the abbreviation

$$\sum_{\lambda=0}^{m\;*} := \sum_{\substack{\lambda=0 \\ m+\lambda \text{ even}}}^{m} \tag{9}$$

Now we can derive a presentation of the Radon transform of a function in the null space.

Theorem 2.2. *Let $f \in \mathcal{N}(\mathscr{A}_p)$ where the directions are mutually different. Then*

$$\mathbf{R}f(\omega,s) = w_1(s)\sum_{m=p}^{\infty} U_m(s)q_m(\omega) \tag{10}$$

with q_m a polynomial of degree $\le m$ in ω, even (odd) for m even (odd) and

$$q_m(\omega) = 0 \text{ for all } \omega \in \mathscr{A}_p\ . \tag{11}$$

Proof. The polynomials q_m in the representation of $\mathbf{R}f$ have p zeroes, hence with the parity condition the polynomials up to order $p-1$ vanish identically. □

For the representation of the functions in the null space themselves we use the inverse Radon transform of the Chebyshev polynomials and the trigonometric functions as given in [9, 36, 27]. Notice that when switching from the notion of spherical harmonics Y_m with $m \ge 0$ and for $m > 0$ consisting of two linear independent elements to their representation as functions we either have $e^{\pm\imath m\varphi}$ or $\sin m\varphi$ and $\cos m\varphi$.

Theorem 2.3. *Let $Q_{m,\lambda}^{\nu,N}$ be the Zernike polynomials, computable as*

$$Q_{m,\lambda}^{\nu,N}(r) = r^\lambda P_{\frac{m-\lambda}{2}}^{(\nu-\frac{N}{2},\lambda+\frac{N}{2}-1)}(2r^2-1) \tag{12}$$

with the Jacobi polynomials $P_n^{(\alpha,\beta)}$ *for* $m+\lambda$ *even and* $0 \le \lambda \le m$. *Then*

$$\mathbf{R}Q_{m,\lambda}^{1,2} Y_\lambda = \frac{2}{m+1} w_1 U_m Y_\lambda \text{ for } m \ge 0 \text{ and } 0 \le \lambda \le m \text{ with } m+\lambda \text{ even} \tag{13}$$

where Y_λ *are the spherical harmonics of order* λ *i.e.,* $Y_\lambda(\omega(\varphi)) = \exp(\pm\imath\lambda\varphi)$.

When normalizing both sides then one gets the singular values of the Radon transform as mapping from $\mathbf{R} : L_2(\Omega_2) \mapsto L_2(C_2, w_1^{-1})$ with singular values $\sigma_{m,\lambda} = O(m^{-1/2})$ independent of λ, see e.g. [9, 32, 38]

Theorem 2.4. *The Radon transform*

$$\mathbf{R} : L_2(\Omega_2) \to L_2(C_2, w_1^{-1})$$

has the singular value decomposition

$$\{(v_{m\lambda}, u_{m\lambda}; \sigma_{m\lambda})\} \quad : \quad m \ge 0,\ \lambda \in \mathbb{N}_0\ ,\ m+\lambda \text{ even}$$

with

$$v_{m\lambda}(x) = \begin{cases} \sqrt{\frac{m+1}{\pi}} Q_{m\lambda}^{1,2} Y_\lambda(\frac{x}{|x|}), & |x| \le 1 \\ 0, & |x| > 1, \end{cases}$$

$$u_{m\lambda}(\omega, s) = \begin{cases} \frac{1}{\pi} w_1(s) U_m(s) Y_\lambda(\omega), & |s| \le 1 \\ 0, & |s| > 1 \end{cases}$$

and

$$\sigma_{m\lambda} = \sigma_m = 2\sqrt{\frac{\pi}{m+1}}$$

Combining the last theorems we get a representation of the functions in the null space.

Theorem 2.5. *Let* $f \in \mathscr{N}(\mathscr{A}_p)$ *for mutually different directions. Then*

$$f(r\omega) = \sum_{m=p}^{\infty} \sum_{\lambda=0}^{m\ *} c_\lambda^m Y_\lambda(\omega) Q_{m,\lambda}^{1,2}(r) \tag{14}$$

where

$$\sum_{\lambda=0}^{m\ *} c_\lambda^m Y_\lambda(\omega) = 0$$

for $\omega \in \mathscr{A}_p$.

Consequently the expansion of a function in the null space $\mathscr{N}(\mathscr{A}_p)$ starts with a polynomial of degree p.

Theorem 2.6. *Let $\mathscr{A}_p$ consist of p mutually different directions and let*

$$\Pi_{p-1} = \{\textit{polynomials of total degree } < p\}$$

Then a function in Π_{p-1} is uniquely determined by the Radon transform on $\mathscr{A}_p$.

Remark 2.7. *The result easily follows from Theorem 2.5. Of course the restriction of the reconstruction problem to polynomials is quite unnatural. But see the discussion in Section 3.2.*

Remark 2.8. *For the uniqueness result the distribution of the directions plays no role, but for the stability it is essential, as the limited angle case shows. See [10] for the proof that the smallest singular values decay exponentially and [31] for the analysis of the dichotomy of the singular values: some of them, where the number depends on the size of the given range, are of the same order as for the full range problem, and only the rest decays exponentially, which means that some parts of the solution can be stably recovered. The artifacts produced by this incomplete data problem are very nicely and convincingly described by Frikel and Quinto, [13] using wavefront sets. Here iterative methods may be an alternative, see Herman, [19] or Jiang and Wang, [20]. Introducing additional information is also a possibility as Vogelgesang and Schorr, [49] show.*

2.2 Equidistributed Directions in Two Dimensions

As the original situation in computed tomography was modeled by the Radon transform, mostly with equidistributed directions, this case was often studied in the literature. Here we call the directions $\omega_j = (\cos\varphi_j, \sin\varphi_j)^\top$ equidistributed when

$$\varphi_j = (j-1)\frac{\pi}{p} \text{ for } j = 1, \cdots, p\,. \tag{15}$$

Simple examples for ghosts were given by Shepp-Kruskal [46], here generalized from their case $p = 4$ to arbitrary p, as

$$f(r\omega(\varphi)) = \begin{cases} \operatorname{sign}(\sin(p\varphi)), & r < 1 \\ 0, & r > 1 \end{cases} \tag{16}$$

and by Herman [19] as

$$f(r\omega(\varphi)) = \begin{cases} \sin(\frac{\pi r}{\delta})\sin(p\varphi), & 0 \le r \le \delta \\ 0, & r > \delta \end{cases} \tag{17}$$

Without difficulties those functions are recognized as ghosts and hence do not really

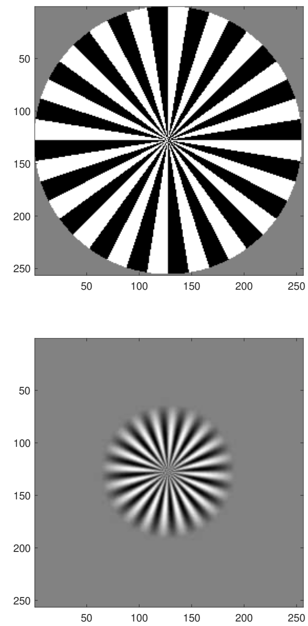

Fig. 2: Simple examples of functions in the null space of the Radon transform for 20 directions of Shepp-Kruskal [46](top) and Herman [19] (bottom).

disturb the reconstructions. In order to construct ghosts that are not as easily recognizable as such, we consider the results from Theorem 2.2. The polynomials q_m from (8) with the zeroes φ_j from (15) can be represented as

$$q_m(\omega(\varphi)) = \sin p\varphi \quad \widetilde{q_{m-p}}(\omega(\varphi)) \tag{18}$$

where $\widetilde{q_{m-p}}$ is an even polynomial of degree $m-p$ for $m \geq p$; i.e.,

$$q_m(\omega(\varphi)) = \sin(p\varphi) \sum_{j=0}^{(m-p)/2} \left(\alpha_{2j}^m \cos(2j\varphi) + \beta_{2j}^m \sin(2j\varphi)\right) \tag{19}$$

Consequently the first term in the expansion of the functions in the null space is

$$g(\omega, s) = c \sin p\varphi \cdot w_1(s) U_p(s)$$

for $c \in \mathbb{R}$ and hence the function itself is

$$f(r\omega(\varphi)) = c \sin p\varphi \cdot r^p \tag{20}$$

This function is depicted in Fig. 1.

With this representation we are now able to construct more realistic ghosts. To this end we project the characteristic function χ_ρ of a ball around 0 with radius ρ on the null space $\mathscr{N}(\mathscr{A}_p)$. As noted in Lemma 2.1 this is no limitation as shifted and dilated versions of those functions are also in the null space.
We have to compute the L_2 scalar product of the characteristic function χ with the basis of the null space. Using the normalized functions from the singular value decomposition, the relation

$$v_{m\lambda} = \frac{1}{\sigma_m^2}\mathbf{R}^*\mathbf{R}v_{m\lambda}$$

and that the Radon transform of the characteristic function of the ball around 0 with radius ρ is

$$\mathbf{R}\chi_\rho(\omega,s) = \begin{cases} 2\sqrt{\rho^2 - s^2}, & s < \rho \\ 0, & s > \rho \end{cases}$$

we compute, remembering the weighted scalar product in $L_2(C_2, w_1^{-1})$,

$$c_m = \frac{4}{\pi}\int_0^\rho U_m(s)\sqrt{\rho^2 - s^2}ds\int_0^{2\pi} q_m(\omega(\varphi))d\varphi$$

for q_m from (19). Because of the orthogonality of the trigonometric functions the only non zero term appears for $2j = p$ and $\sin p\varphi$. Hence this approach results in a nontrivial example only if the number of directions is even. With

$$\sin^2(p\varphi) = \frac{1}{2}(1 - \sin(2p\varphi))$$

we get the representation for the Radon transform of the ghosts as

$$\mathbf{R}f = \sum_{m=p}^{\infty} c_m\left(u_{2m,0} - u_{2m,2p}\right) \tag{21}$$

and hence for the function itself

$$f = \sum_{m=p}^{\infty} \frac{c_m}{\sigma_m}\left(v_{2m,0} - v_{2m,2p}\right) \tag{22}$$

or explicitely

$$f(r\omega(\varphi)) = \sum_{m=p}^{\infty} \frac{c_m(m+1)}{2\pi^{3/2}}\left(P_m^{(0,0)}(2r^2-1) - r^{2p}P_{m-p}^{(0,2p)}(2r^2-1)\sin(2p\varphi)\right). \tag{23}$$

In Fig. 3 and 4 we present ghosts generated by projecting the characteristic functions of the ball around 0 with radius $\rho = 0.01$ and $\rho = 0.1$ for 20 equidistributed directions. Observe that the constant functions are not in the null space, hence essentially the boundary of the characteristic functions with some wiggles appear.

We observe that the quality of the examples are, due to increased computer power, much superior to those from almost 40 years ago in the original paper [26].

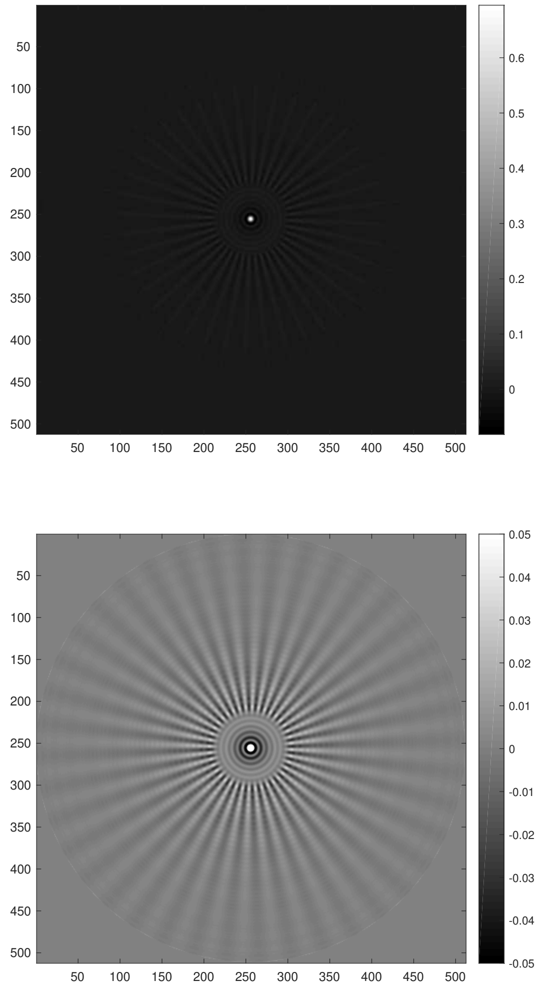

Fig. 3: Projection of the characteristic function of a ball around 0 with radius 0.01 on the null space for 20 equidistributed directions. Top: Full display window such that the largest value (0.695) presented as white, Bottom: display window truncated such that everything larger than 0.05 is white in order to show the wiggles outside the center.

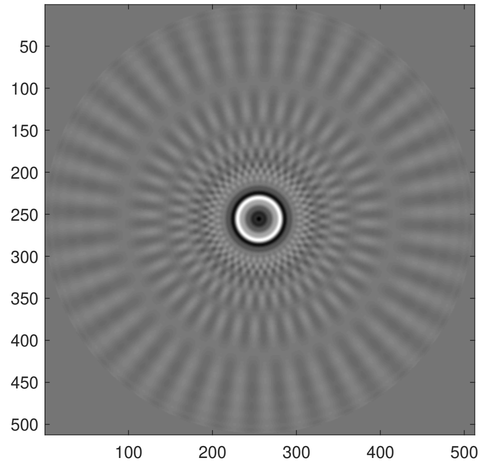

Fig. 4: Projection of the characteristic function of a ball around 0 with radius 0.1 on the null space, full display window. Compare to Fig 3 TOP.

2.3 Higher Dimensional Case

Three dimensional Radon transforms appeared in the first magnetic resonance imaging scanners, in those days called nuclear magnetic zeugmatograpy, [37]. Even four dimensional Radon transforms are the mathematical model in electron paramagnetic resonance imaging (EPRI) [24] where in addition to up to three spatial dimensions a further spectral dimension is measured. Besides it is of course of theoretical interest to analyze the null space.

In N dimensions the Radon transform is defined by integrating over $N-1$ dimensional hyperplanes

$$\mathbf{R}f(\omega,s) = \int_{\mathbb{R}^N} f(x)\delta(s - x^\top \omega)dx \,.$$

In principle we can start again from the consistency conditions. The integration of the Radon transform of a function against a polynomial in s results in a polynomial in the directions. The first selection we have to make is the polynomials. When we start with a radial symmetric weight W on the space of the searched-for functions, the weight in the image space is $w = \mathbf{R}W$ which is a function of the distance s. For different pairs of weights and the corresponding polynomials inversion formulas and consequently singular value decompositions are given in [30].
For the sake of simplicity we restrict the considerations in the following on the weight

$W \equiv 1$ resulting in the weight $w_{N/2}$ in N dimensions where

$$w_{N/2}(s) = (1-s^2)^{(N-1)/2} .$$

The corresponding orthogonal polynomials are the Gegenbauer or ultraspherical polynomials, denoted by $C_m^{N/2}$. Hence we consider the Radon transform as continuous mapping

$$\mathbf{R} : L_2(\Omega_N) \to L_2(C_N, w_{N/2}^{-1})$$

with Ω_N the unit ball in $\mathbb{R}^N$ and $C_N = S^{N-1} \times [-1,1]$. Then the polynomials with respect to s in the range of the Radon transform are the Gegenbauer polynomials $C_m^{N/2}$ which are orthogonal on $[-1,1]$ with respect to the weight $w_{N/2}(s) = \left(1-s^2\right)^{(N-1)/2}$.

The polynomials in the directions are represented using spherical harmonics, see e.g. [43]. The spherical harmonics form an orthonormal system on $L_2(S^{N-1})$. Hence the functions in the range of the Radon transform can be represented as

$$\mathbf{R}f(\omega,s) = w_{N/2}(s) \sum_{m=0}^{\infty} C_m^{N/2}(s) q_m(\omega) \tag{24}$$

where again

$$q_m(\omega) = \sum_{\lambda=0}^{m} \sum_{\mu=1}^{* M(N,\lambda)} c_{\lambda,\mu}^m Y_{\lambda,\mu}(\omega) \tag{25}$$

and

$$M(N,\lambda) = \frac{(2\lambda+N-2)(N+\lambda-3)!}{\lambda!(N-2)!} = O(\lambda^{N-2})$$

the dimension of the spherical harmonics fo degree λ in $\mathbb{R}^N$

The number of linear independent polynomials for fixed degree is increasing for increasing dimension.

With this representation we can transfer the derivation of the functions in the null space from the two- to higher dimensional case. We again define the set of directions as

$$\mathscr{A}_p = \{\omega_1, \dots \omega_p\} \subset S^{N-1} , \tag{26}$$

a set of p distinct directions. The distribution of the directions on the unit sphere again plays no role for the uniqueness questions, whereas for the stability of the inversion it is crucial.

Next we consider the null space of the Radon transform for those directions

$$\mathscr{N}_p = \mathscr{N}(\mathscr{A}_p) = \{f \in L_2(\Omega_N) : \mathbf{R}f(\omega,s) = 0 \text{ for all } \omega \in \mathscr{A}_p \text{ and almost all } s \in [-1,1]\} \tag{27}$$

In order to be in the null space the expansion has to fulfill the condition that

$$q_m(\omega) = 0 \text{ for all } m \geq 0 \text{ and for all } \omega \in \mathscr{A}_p \text{ and almost all } s \in [-1,1].$$

But there is an essential difference in the higher dimensional case. In the 2D case it suffices that the directions are mutually distinct in order to conclude that for p directions the polynomials up to order $p-1$ vanish identically. As already known from the algebraic polynomials this does not suffice in higher dimensions. For example a polynomial of degree 1 in two dimensions has three coefficients, namely for 1, x_1 and x_2. If such a polynomial has 3 zeroes this does not imply that it vanishes identically. Only if the zeroes do not lie on a line one can conclude that the polynomial vanishes identically. To express this rigorously we define, following [30],

$$\mathscr{P}_m = \operatorname{span}\{Y_{\lambda\mu} \;:\; 0 \le \lambda \le m \;,\; \lambda + m \text{ even } \;,\; 1 \le \mu \le M(N,\lambda)\} \tag{28}$$

with

$$\dim \mathscr{P}_m = \binom{m+N-1}{N-1} \,.$$

Theorem 2.9. *([30],Theorem 5.2) Let $n > 0$ and $p \ge \dim \mathscr{P}_{n-1}$. Let $\mathscr{A}_p$ not be contained in an algebraic variety of degree $< n$; i.e., there is no $q \in \mathscr{P}_{n-1}$, $q \not\equiv 0$ with $q(\omega) = 0$ for all $\omega \in \mathscr{A}_p$. Let $f \in \mathscr{N}_{A_p}$. Then*

$$\mathbf{R}f = w_{N/2} \sum_{m=n}^{\infty} C_m^{N/2} q_m(\omega) \tag{29}$$

where $q_m \in \mathscr{P}_m$ with $q_m(\omega) = 0$ for all $\omega \in \mathscr{A}_p$.

Remark 2.10. *The condition in the above Theorem concerning the distribution of the directions not lying in an algebraic variety of degree $< n$ was later called by Natterer [38] n-resolving.*

The reasoning for this expression becomes clear in the next section. It gives also conditions for optimally distributing the directions, see [28]. Again we can conclude a uniqueness result.

Theorem 2.11. *Let the conditions from Theorem 2.9 be fulfilled. Then the reconstruction problem in Π_{n-1} is unique.*

Finally we mention the functions in the null space themselves. For that we need the inverse Radon transform of the $w_{N/2} C_m^{N/2} Y_{\lambda\mu}$, see Corollary 3.2 in [30].

$$\mathbf{R}^{-1}\left(w_{N/2} C_m^{N/2} Y_{\lambda\mu}\right)\left(r\omega\right) = c(N,m) r^{\lambda} P_{(m-\lambda)/2}^{(0,\lambda+N/2-1)}(2r^2-1) Y_{\lambda\mu}(\omega) \tag{30}$$

with the constant

$$c(N,m) = \pi^{1-N/2} 2^{1-n} \Gamma(m+N)/[\Gamma(m+1)\Gamma(N/2)] \,.$$

The result for arbitrary weights W_ν is contained in Theorem 3.1 in [30]. The prove uses the projection theorem for the Radon transform and special functions leading to Bessel functions, see next section and the Funk-Hecke theorem, [43], and integrals over Bessel functions of the Weber-Schafheitlin type, for the Hankel transform.

3 Resolution

The oscillatory character of the functions in the null space advise to analyze their spectrum. This is of importance for the study of the inversion algorithms where mostly high frequency components of the data are damped or even eliminated. Consequently we start with the Fourier transform of the functions in the following. It is defined as

$$\mathscr{F}_N f(\xi) = \hat{f}(\xi) = (2\pi)^{-N/2} \int_{\mathbb{R}^N} f(x) \exp(-\imath x^\top \xi) dx \,. \tag{31}$$

A very helpful relation is the so-called

Theorem 3.1. *(Projection Theorem) For fixed direction $\omega \in S^{N-1}$ we have*

$$\mathscr{F}_1 \mathbf{R} f(\omega, \sigma) = (2\pi)^{(N-1)/2} \mathscr{F}_N f(\sigma\omega) \;, \sigma \in \mathbb{R} \tag{32}$$

This relation is also known as central slice theorem. For compactly supported functions f it simply can be interpreted as applying the dual operator of the Radon transform to the suitably truncated exponential function. Otherwise it is a mere change of coordinates. The Paley-Wiener theorem states that the Fourier transform of a compactly supported L_2 function is analytic.

Now we can determine the Fourier transform of the ghosts. Due to the projection theorem it is sufficient to know the Fourier transform of $w_{N/2} C_m^{N/2}$.

Theorem 3.2. *([29],Theorem 2.1) Let the conditions of Theorem 2.9 be fulfilled and let $f \in \mathscr{N}(\mathscr{A}_p)$. Then*

$$\mathscr{F}_N f(\sigma\omega) = \sigma^{-N/2} \sum_{m=n}^{\infty} \imath^m J_{m+N/2}(\sigma) q_m(\omega) \tag{33}$$

where J_k is the Bessel function of the first kind and order k and $q_m(\omega) = 0$ for all $\omega \in \mathscr{A}_p$.

This means that the expansion of the Fourier transform of the ghosts starts with the Bessel function of oder $m + N/2$. Following Debye's formula the Bessel functions are small when the argument is smaller than the index, see e.g. [2] 9.3.7, we interpret this that the Fourier transform of the ghosts is 'small' when the argument is smaller than $n+N/2$. Of course 'small' is not an adequate expression here, because multiplying a ghost by any number still remains a ghost. So, in the following we study the relative size of the ghost inside that mentioned ball compared to the whole energy.

Definition 3.3. *If the Fourier transform of a function is compactly supported, say in $[-b,b]^N$ or in $V(0,b)$, the ball around 0 with radius b, then we call the function b band limited.*

The above statement about the analyticity of the Fourier transform of a compactly supported function implies that there is no function, besides 0, that is both compactly supported and band limited. As the functions we consider in tomography are always compactly supported they cannot be band limited. A way out is the concept of essentially band limited functions introduced by Natterer, [38].
A strong result about band limited function is the sampling theorem, see Shannon [44].

Theorem 3.4. *(Sampling Theorem) Let f be b band limited, and let $h \leq \pi/b$. Then f is uniquely determined by the values $f(hk)$, $k \in \mathbb{Z}^N$.The size π/b is called the Nyquist rate.*

In L_2 the function can be represented by the so-called sinc - series. For a simple and elegant proof see [38], page 56.

3.1 The General Case

Logan [25] proved in two dimensions a theorem that very roughly states that the knowledge of the full projections in each of p directions is sufficient to reconstruct the searched-for function f up to but not beyond bandwidth p. A little more precisely, Logan shows that a function in $L_2(\Omega)$ is of essential bandwidth $p(1-\varepsilon)$, it can be essentially reconstructed from any p views. On the other band, he proves that there exist functions of essential bandwidth $p(1+\varepsilon)$ which project to zero in any p given directions, as reformulated by Shepp-Kruskal [46].
Correct filtering of the functions in the reconstruction process can also reduce the influence of the ghosts. As an incentive for the following considerations we consider a substantial simplification of the proof of Logan [25] which sheds light upon the essential properties of the ghosts and both improves the results and generalizes them to higher dimensions. The presentation follows Louis [29].
We study the behavior of the spectrum of the ghosts and show that they are high-frequency functions. To this end we compare the energy of their Fourier transform lying inside a ball around 0 of radius c with their total energy. Remembering the condition

from Theorem 2.9, where the number of given projections, p, is related to the integer n by $p = \dim \mathscr{P}_{n-1}$ we define

Definition 3.5. *The quotient between the energy of a function inside a ball* $V(0,c)$ *around* 0 *with radius c and the total energy is defined as*

$$\mu_n(c) = \sup\{\frac{\|\hat{f}\|^2_{L_2((V(0,c))}}{\|f\|^2_{L_2(\mathbb{R}^N)}} \ : \ f \in \mathscr{N}(\mathscr{A}_p)\} \tag{34}$$

where we have used Parseval's relation that $\|f\|_{L_2(\mathbb{R}^N)} = \|\hat{f}\|_{L_2(\mathbb{R}^N)}$.
If this size is small then most of the power of the ghosts is lying outside $V(O,c)$ and a cut-off of the frequencies larger than c prevents these functions form disturbing the reconstructions.

In order to determine $\mu_n(c)$ we perform the following steps, always assuming that the conditions from Theorem 2.9 are fulfilled. For details see [29] and for helpful tables of integrals see [2, 15].
We use the *Hankel* transform defined as

$$\mathscr{H}_\nu^N u(s) = s^{1-N/2} \int_0^\infty u(\sigma)\sigma^{N/2} J_{\nu+N/2-1}(s\sigma)d\sigma$$

where J_ν is the Bessel function of the first kind of order ν. The Fourier transform of a function $f(r\omega) = u(r)Y_{\lambda\mu}(\omega)$ can be computed with the help of the Hankel transform to

$$\mathscr{F}_N f(\sigma\omega) = \mathscr{H}_\lambda^N u(\sigma) Y_{\lambda\mu}(\omega) \ .$$

In a first step we connect $\mu_n(c)$ with the largest eigenvalue in modulus $\gamma_{n+N/2-1,0}(c)$ of an integral operator with kernel

$$K(s,\sigma) = (s\sigma)^{1/2} J_{n+N/2-1}(cs\sigma) \ , = \leq s,\sigma \leq 1$$

and prove

$$\mu_n(c) = c^2 \gamma^2_{n+N/2-1,0}(c)$$

Classical results for the eigenvalues of integral operators, for special functions and a result of Slepian [47] lead to the mentioned equality.

It remains to calculate lower and upper bounds for $\mu_n(c)$.

Theorem 3.6. *Let the conditions from Theorem 2.9 be fulfilled. Let* c_n *be a sequence such that*

$$c_n \leq (n+N/2-1) - \alpha(n+N/2-1)^\beta \tag{35}$$

with $\alpha > 0$ *and* $\frac{1}{3} < \beta > 1$. *Then*

$$\lim_{n\to\infty} \mu_n(c_n) = 0 \ .$$

For a lower bound we find

Theorem 3.7. *Let the conditions from Theorem 2.9 be fulfilled. Let c_n be a sequence such that*

$$c_n \geq (n+N/2-1)+\alpha(n+N/2-1)^{\beta} \tag{36}$$

with $\alpha > 0$ and $\beta > 1/2$. Then

$$\lim_{n\to\infty} \mu_n(c_n) = 1 \, .$$

Following the sampling theorem objects of the size $(n+N/2-1)^{-1}$ or larger can reliably be recovered in the reconstruction if p directions are used such that condition from Theorem 2.9 is fulfilled. Smaller details may be disturbed by functions in the null space. Shannon's sampling theorem also states that a sampling distance $\Delta s = \pi(n+N/2-1)^{-1}$ is sufficient if the frequencies larger than $n+N/2-1$ are filtered. Therefore we can replace the demand for complete projections by the condition that the data are sampled with a rate less than or equal to Δs.
Note that the cut-off frequency in $\mathbb{R}^N$ is $n+N/2-1$ which in the case $N=2$ coincides with the number p of given directions. In general it is $O(p^{1/(N-1)})$. The example in Fig 3 shows what happens is one tries to reconstruct disregarding these resolution limits: the ghost is too small to fulfill the limits of the above theory.

3.2 Restrictions on the Functions Space for Improving the Resolution, Compressive Sensing and Deep Learning

Already in 1974 Marr [36] published a uniqueness result when assuming the searched-for function is a polynomial of highest degree depending on the number of data. He even used a finite number of rays in a fan-beam geometry. Shepp-Kruskal [46] honorably mentioned that result, but criticized at the same time that polynomials are quite unnatural as space for the images to be reconstructed. Especially the pictures in tomography are far from being smooth, they often are considered at most to be piecewise smooth. Hence, the same criticism includes the Theorems 2.6 and 2.11. Of course everyone knows polynomials and it is clear that they are not the natural candidates.

More success was reached when the well-known polynomials are replaced by something less precise: the condition that was studied is sparsity, sparsity in a suitable basis. So it is up to the user to find the right basis or dictionary. When the user fails it is then not the problem of the method. The method was named compressive sensing, see e.g. [6] or [8]. A statement that one can beat Shannon or Nyquist can be considered as dubious because the success is based on the restriction of the set of functions to be reconstructed. For a constant function one measurement suffices and that cannot be improved, if exact data are considered.

In recent times deep learning is the magic method. Neural networks are trained for special tasks. Neural networks were in use decades ago, but they were criticized that the layers were unphysical. Now one increases largely the number of layers with simple operations, like convolutions, which leads to the expression deep, and one adds a miraculous nonlinear map in each step to be found by the user. With all the effort going in this direction by highly qualified colleagues success is to be expected, see e.g. [3, 4]

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