Numerical Inversion of the Spherical Radon Transform

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Spherical Radon transform

It is defined by

\[ Rf(\theta, \varphi) = \frac{1}{2\pi} \int_{[0,2\pi]} \int_{S^2} f(\omega) \sin^2 \gamma d\omega d\gamma, \]

with the surface area measure on the subsphere \(S^2 \cap \eta^2\), \(L^2(S^2)\) even functions in \(L^2\) over the sphere \(S^2\).

Properties of the transform

The spherical Radon transform
- is a linear, continuous operator,
- is self-adjoint, i.e. \(R^* = R\),
- is a bijection from \(C^0_c(S^2)\) to \(C^0_c(S^2)\),
- commutes with rotations, i.e. for all \(\phi \in SO_3\) we have

\[ R(T_{\phi} f) = T_{\phi} Rf \]

with \(T_{\phi} f(x) = f(\phi^{-1} x)\).

Desired mollifiers

- The spherical Radon transform is only defined for even functions, so only even mollifiers are applicable.
- Due to the rotational invariance, it suffices to declare \(e_\phi\) for an arbitrary reconstruction point \(x\). We choose the third unit vector \(e_3\) and consider mollifiers which only depend on the polar angle \(\theta\).

Two interesting examples of mollifiers can be seen in the following figures as functions of the polar angle \(\theta\).

- Figure 1: Characteristic mollifier.
- Figure 2: Gauss mollifier.

Reconstruction kernels

- Example 1:

\[ \phi_\theta(x, y) = e^{\gamma(y)} \begin{cases} 1, & \text{if } |y| < \sin \gamma, \\ \frac{1}{2 \sin \gamma} & \text{else}. \end{cases} \]

The associated mollifier can now be calculated via \(\tilde{R} \phi_\theta = e_\gamma\).

- Example 2:

\[ \phi_{\gamma}(x, y) = \beta \frac{1}{\sqrt{\sin^2 \gamma - |y|^2}}, \]

for \(\gamma \geq 0\).

Analytic inversion formula

For \(g \in C^0_c(S^2)\) the following inversion formula holds

\[ g(\omega) = \frac{1}{2\pi} \int_{[0,2\pi]} \int_{S^2} R_{\gamma} \phi_{\gamma}(\omega, y) d\omega d\gamma. \]

Problem: This formula cannot be used numerically.

Approximate Inverse

- We are looking for the solution of \(Rf = g\).
- Instead of \(f\), we reconstruct an approximate version \(f_\gamma\) with the property

\[ f_\gamma \rightarrow f \quad \text{as} \quad \gamma \rightarrow 0. \]

- Solve the auxiliary problem

\[ R^* \tilde{\phi}_\gamma(x, \cdot) = R \phi_{\gamma}(x, \cdot) + e_\gamma(x, \cdot), \]

with chosen mollifier \(e_\gamma\) and reconstruction kernel \(\tilde{\phi}_\gamma\). Then define

\[ f_\gamma(x, \cdot) = (f, e_\gamma(x, \cdot)) = (g, \phi_{\gamma}(x, \cdot)). \]

Advantages
- Built in regularization.
- Auxiliary problem is solvable independent of the \(g\).

This leads to an efficient and stable algorithm.

Numerical integration over the sphere

We use the formula

\[ \int_{S^2} f(\omega) d\omega = 2 \sum_{l=0}^{\ell_{\max}} \sum_{m=-l}^{l} g(l, \ell, \ell) M(4\pi l) \frac{M(4\pi l)}{4\pi l} \]

with \(g(l, \ell, \ell) = \int_{S^2} f(\omega) \cos(\ell \varphi) \sin(\ell \theta) \sin(\ell \theta) \sin(\ell \theta) \cos(\ell \theta)\) for an even function \(f\).

We discretize the sphere as seen in the figure on the right hand and weight the data with the corresponding surface area.

References