3D Imaging in Cone Beam Vector Field Tomography

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Aims

Vector field tomography (VFT) deals with the problem of reconstructing a vector field, e.g. a velocity field of an incompressible, moving fluid, from line integrals of projections of the field. The integral data can be measured using ultrasound signals when we assume that the Doppler shift of the frequency is approximately proportional to the velocity of the particle in the fluid which causes the shift. This is a reasonable assumption when the particle velocity is significantly smaller than the speed of sound within the medium under consideration. Mathematically we have to invert the vectorial cone beam transform.

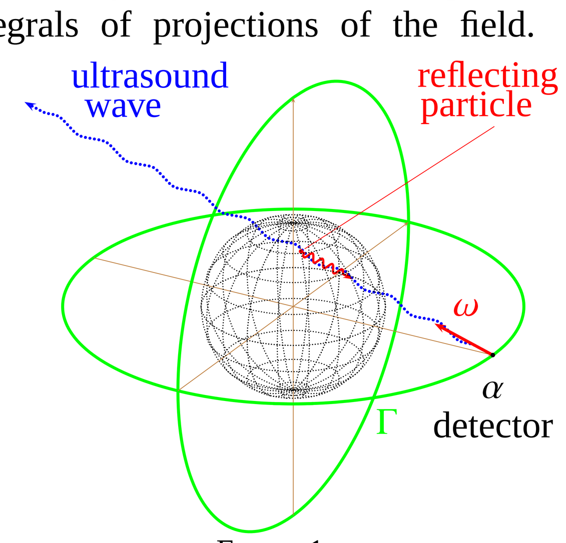


Figure 1
Measurement setup with two orthogonal circles as scanning curve Γ

VFT has applications in photoelasticity, oceanography, non-destructive testing and medical imaging where we may think of tumor detection by reconstructing and visualizing blood flow which is known to be more irregular and more intense around tumors than in normal tissue.

Cone beam transform

It is defined for a tensor field of rank m by

$$\mathbf{D}_m\mathbf{f}(\alpha,\omega) = \int_0^\infty \langle \mathbf{f}(\alpha+t\omega), \omega^m\rangle \, dt = \int_0^\infty \mathbf{f}_{i_1\cdots i_m}(\alpha+t\omega)\,\omega^{i_1}\cdots\omega^{i_m}\,dt \,, \tag{1}$$

where $\alpha \in \Gamma$ is a source point of the scanning curve $\Gamma \subset \mathbb{R}^n \setminus \overline{\Omega}$ which surrounds the object Ω, $\omega \in S^2$ is the unit vector of direction of the line and $\mathbf{f}$ is a tensor field of rank m with compact support in the open domain Ω. In (1) we use Einstein's summation rule, that means we sum up over equal indices i_j, where $1 \le i_j \le n$.

$m = 0$: well-known scalar cone beam transform of a scalar field f

$$\mathbf{D}_0 f(\alpha,\omega) = \int_0^\infty f(\alpha+t\omega)\,dt \,. \tag{2}$$

$m = 1$: cone beam transform of a vector field $\mathbf{f}$

$$\mathbf{D}_1\mathbf{f}(\alpha,\omega) = \int_0^\infty \langle \mathbf{f}(\alpha+t\omega), \omega\rangle\,dt \,. \tag{3}$$

Hence the mathematical problem of 3D cone beam VFT consists of inverting $\mathbf{Df} = g$ for given measurements g. The method of **approximate inverse** introduced by Louis and Maass [1] delivers a mathematical framework for coping with inverse problems in an efficient way.

Mathematical Properties of D

Let $\mathcal{S}^m$ be the set of all symmetric, covariant tensors of rank m

$$\mathcal{S}^m = \left\{\mathbf{f} = f_{i_1\cdots i_m} dx^{i_1}\otimes\cdots\otimes dx^{i_m}\right\} \text{ for } 1 \le i_j \le n,\ 1 \le j \le m.$$

Tensors of rank 1 are vectors in $\mathbb{R}^n$. A mapping

$$x \mapsto \mathbf{f}(x) = f_{i_1\cdots i_m}(x) dx^{i_1}\otimes\cdots\otimes dx^{i_m} \qquad x \in \Omega$$

represents a symmetric, covariant tensor field. Tensor fields of rank 0 are scalar functions $f(x)$, tensor fields of rank 1 are vector fields $\mathbf{f}(x)$ in $\mathbb{R}^n$. The space of square integrable, symmetric, covariant tensor fields of rank m in $\Omega \subset \mathbb{R}^n$ is denoted by

$$L^2(\Omega,\mathcal{S}^m) := \left\{\mathbf{f}\in\mathcal{S}^m : \|f\|_{L^2} = \langle\mathbf{f},\mathbf{f}\rangle_{L^2}^{1/2} < \infty\right\} \,,$$

where the L^2-inner product of two tensor fields is given as

$$\langle\mathbf{f},\mathbf{g}\rangle_{L^2} = \int_{\Omega^n} f_{i_1\cdots i_m}(x) g^{i_1\cdots i_m}(x)\,dx \,.$$

Theorem 1.
Let $\Omega^n := \{x\in\mathbb{R}^n : |x| < 1\}$ with $\partial\Omega^n = S^{n-1}$. The mapping $\mathbf{D} : L^2(\Omega^n,\mathcal{S}^m) \to L^2\left(\Gamma\times S^{n-1}\right)$ is linear and bounded, if

$$\int_\Gamma (|\alpha|-1)^{1-n}\,d\alpha < \infty \,.$$

The **adjoint** (backprojection) $\mathbf{D}^* : L^2\left(\Gamma\times S^{n-1}\right) \to L^2(\Omega^n,\mathcal{S}^m)$ is given by

$$\mathbf{D}^* g(x) = \int_\Gamma \left\{|x-\alpha|^{1-n-m}\, g\left(\frac{x-\alpha}{|x-\alpha|}\right)(x-\alpha)^m\right\} d\alpha \,,$$

where $(x-\alpha)^m = \underbrace{(x-\alpha)\otimes\ldots\otimes(x-\alpha)}_{\text{m times}} \in \mathcal{S}^m$.

$m = 0$, $n = 3$: cone beam transform with corresponding backprojection operator of scalar fields, thoroughly investigated in 3D CT
$m = 1$, $n = 3$: 3D cone beam VFT, the backprojection reads as

$$\mathbf{D}^* g(x) = \int_\Gamma |x-\alpha|^{-2}\, g\left(\frac{x-\alpha}{|x-\alpha|}\right)\frac{x-\alpha}{|x-\alpha|}\,d\alpha \,. \tag{4}$$

One of the crucial tools when computing reconstruction kernels in scalar cone beam tomography is the **formula of Grangeat** [3]. We proved a generalization of that formula which is valid for any tensor field of rank m in n dimensions.

Theorem 2. (Schuster [4] based on Hamaker et al. [5])
Assume $n \ge 2$ and $\mathbf{f} \in C_0^{(n-2)}(\Omega^n,\mathcal{S}^m)$. Then,

$$\frac{\partial^{(n-2)}}{\partial s^{(n-2)}}\mathbf{R} f_\alpha^m(\omega,\langle\alpha,\omega\rangle) = (-1)^{(n-2)} \int_{S^{n-1}} \mathbf{Df}(\alpha,\theta)\,\delta^{(n-2)}(\langle\omega,\theta\rangle)\,dS(\theta) \,, \tag{5}$$

where $\alpha \in \Gamma$, $\omega \in S^{n-1}$, dS denotes the surface measure on S^{n-1}, $\mathbf{R}$ is the n-dimensional Radon transform and

$$f_\alpha^m(x) = \left\langle\mathbf{f}(x), |x-\alpha|^{-m}(x-\alpha)^m\right\rangle = f_{i_1\ldots i_m}(x)\,|x-\alpha|^{-m}(x-\alpha)^{i_1}\cdots(x-\alpha)^{i_m}$$

is the projection of $\mathbf{f}$ onto $\frac{x-\alpha}{|x-\alpha|}$.

For $m = 1$, $n = 3$ formula (5) reads as

$$\frac{\partial}{\partial s}\mathbf{R} f_\alpha^m(\omega,\langle\alpha,\omega\rangle) = \int_{S^2\cap\{\langle\theta,\omega\rangle=0\}} \left\langle\nabla_y\mathbf{D} f(\alpha, y=\theta),\omega\right\rangle dS(\theta) \,. \tag{6}$$

In the scalar case a solver for $\mathbf{D}$ can be constructed with the help of (5). This is done by inverting the Radon transform $\mathbf{R}$ which is possible if the condition of Tuy-Kirillov is satisfied. It tells that we have full knowledge of $\mathbf{R} f_\alpha^m(\omega,s)$ for all ω, s, if any plane intersecting the object Ω does also have at least one intersection point with the scanning curve Γ and this intersection must be non-transversally. Unfortunately that does not help in case $m \ge 1$, since there the function f_α^m depends on α and hence the object function f_α^m of $\mathbf{R}$ changes with α, see (6). Thus we seek an alternative way of solving $\mathbf{D} f = g$ in case $m = 1$, i.e. for vector fields $\mathbf{f}$.

Approximation of Reconstruction Kernels

The method of **approximate inverse** was established by Louis and Maas [1] in 1990, see also [6], [7], and was successfully applied to several reconstruction problems in non-destructive testing and medical imaging, such as computerized tomography and Doppler tomography. Results for Doppler tomography in the parallel geometry were presented in [8]. In [2] the method was applied to 3D cone beam tomography, i.e. to $\mathbf{D}_0$. We describe this approach and formulate its extension to $\mathbf{D}_1$.
Let $f \in L^2\left(\Omega^3\right)$. The approximate inverse computes a smoothed version of f by convolving f with a mollifier $e_\gamma \in C^\infty\left(\mathbb{R}^3\right)$. A mollifier e_γ is a smooth function with small essential support having the property that

$$f_\gamma(x) := \left(f * e_\gamma\right)(x) \to f(x) \quad \text{as } \gamma\to 0 \,.$$

$*$ denotes the convolution: $\quad (f*h)(x) = \int_{\mathbb{R}^3} f(y-x)\,h(y)\,dy$

Usage of Gaussian kernel for e_γ: $\quad e_\gamma(x) = \frac{\gamma^{-3}}{(2\pi)^{3/2}}\exp\left(-\frac{|x|^2}{2\gamma^2}\right) \qquad (7)$

Provided that we can solve the equation: $\quad \mathbf{D}_0^*\left[v_\gamma(x)\right] = e_\gamma(x-\cdot)$

then:

$$f_\gamma(x) = \left\langle\mathbf{D}_0 f, v_\gamma(x)\right\rangle_{L^2(\Gamma\times S^2)} = \int_\Gamma\int_{S^2} (\mathbf{D}_0 f)(\alpha,\omega)\; v_\gamma(x;\alpha,\omega)\; dS(\omega)\,d\alpha \,,$$

where $v_\gamma(x) = v_\gamma(x;\alpha,\omega) \in L^2\left(\Gamma\times S^2\right)$ for $x\in\Omega$ is called a reconstruction kernel. Hence the method of approximate inverse consists of evaluating inner products of the given data $\mathbf{D}_0 f$ with reconstruction kernels $v_\gamma(x)$, what can be done in an efficient way by using the translation invariance of e_γ.
To apply the method to $\mathbf{D}_1$ and hence to VFT, we construct mollifier fields $\mathbf{E}_\gamma \in L^2\left(\Omega^3,\mathcal{S}^1\right)$ defining

$$\mathbf{E}_\gamma^j(x) := e_\gamma(x)\cdot e_j \,, \qquad j\in\{1,2,3\}$$

where $e_1 = (1,0,0)^\top$, $e_2 = (0,1,0)^\top$, and $e_3 = (0,0,1)^\top$. Using again the Gaussian (7) as mollifier e_γ we have

$$\left(\mathbf{f}_\gamma\right)_j(x) := \left(\mathbf{f} * \mathbf{E}_\gamma^j\right)(x) \to \mathbf{f}_j(x) \quad \text{as } \gamma\to 0$$

for $\mathbf{f}\in L^2\left(\Omega^3,\mathcal{S}^1\right)$. Unfortunately, by now the exact reconstruction kernels $\mathbf{V}_\gamma^j(x)$, i.e. the solutions of $\mathbf{D}_1^*\left[\mathbf{V}_\gamma(x)\right] = \mathbf{E}_\gamma^j(x-\cdot)$ are still unknown.

But the special structure of the mollifier fields $\mathbf{E}_\gamma$ allow for a computation of reconstruction kernels for

$$\mathbf{Pf}(\alpha,\omega) = \int_0^\infty \mathbf{f}(\alpha+t\omega)\,dt$$

with the help of kernels for $\mathbf{D}_0$.

Theorem 3.
Let v_γ be the reconstruction kernel associated to e_γ with respect to $\mathbf{D}_0$, that is

$$\mathbf{D}_0^*\left[v_\gamma(x)\right] = e_\gamma(x-\cdot) \,.$$

Defining $\mathbf{V}_\gamma^j(x;\alpha,\omega) = v_\gamma(x;\alpha,\omega)\cdot e_j \in L^2\left(\Gamma\times S^2,\mathbb{R}^3\right)$ yields

$$\mathbf{P}^*\left[\mathbf{V}_\gamma^j(x)\right] = \mathbf{E}_\gamma^j(x-\cdot) \,,$$

that means $\mathbf{V}_\gamma^j$ is a reconstruction kernel associated to $\mathbf{E}_\gamma^j$ with respect to $\mathbf{P}$. The adjoint $\mathbf{P}^*$ of $\mathbf{P}$ is given as

$$\mathbf{P}^*\mathbf{g}(x) = \int_\Gamma |x-\alpha|^{-2}\,\mathbf{g}\left(\alpha, \frac{x-\alpha}{|x-\alpha|}\right)d\alpha$$

for $\mathbf{g}\in L^2\left(\Gamma\times S^2,\mathbb{R}^3\right)$.

The data $\mathbf{Pf}$ are not known and cannot be computed from $\mathbf{D}_1\mathbf{f}$. Moreover, observing that $\mathbf{D}_1\mathbf{f}(\alpha,\omega) = \langle\mathbf{Pf}(\alpha,\omega),\omega\rangle$ and since $\mathbf{Pf}(\alpha,\omega)\in\mathbb{R}^3$ we get

$$\mathbf{Pf}(\alpha,\omega) = \mathbf{D}_1\mathbf{f}(\alpha,\omega)\,\omega + \lambda_1\left(\alpha,\omega_1^\perp\right)\omega_1^\perp + \lambda_2\left(\alpha,\omega_2^\perp\right)\omega_2^\perp \,,$$

where $\omega_1^\perp, \omega_2^\perp \in S^2$ are such that $\left\{\omega,\omega_1^\perp,\omega_2^\perp\right\}$ is an orthonormal basis of $\mathbb{R}^3$ and λ_1, λ_2 are appropriate coefficients. Thus approximating

$$\mathbf{Pf}(\alpha,\omega) \approx \mathbf{D}_1\mathbf{f}(\alpha,\omega)\,\omega$$

we neglect the parts orthogonal to ω and can apply the method of approximate inverse using the reconstruction kernels $\mathbf{V}_\gamma^j$ for $\mathbf{P}$.

Algorithm for Cone Beam VFT

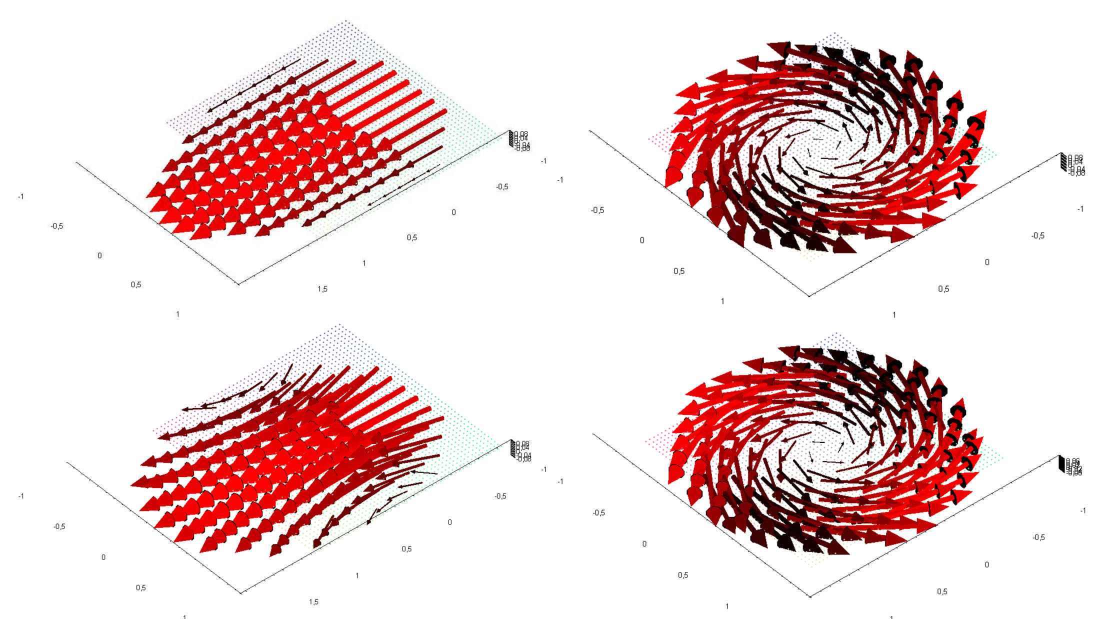
Given : Measured data $\mathbf{D}_1\mathbf{f}(\alpha,\omega)$ for $\alpha\in\Gamma$, $\omega\in S^2$
Output : Approximation $\mathbf{f}_\gamma$ to $\mathbf{f}$
Compute :

- $\mathbf{g}(\alpha,\omega) = \mathbf{D}_1\mathbf{f}(\alpha,\omega)\,\omega$
- $\left(f_\gamma\right)_j(x) = \left\langle\mathbf{g},\mathbf{V}_\gamma^j(x)\right\rangle_{L^2(\Gamma\times S^2,\mathbb{R}^3)} = \int_\Gamma\int_{S^2}\left\langle\mathbf{g}(\alpha,\omega),\mathbf{V}_\gamma^j(x;\alpha,\omega)\right\rangle dS(\omega)\,d\alpha$

for $j\in\{1,2,3\}$.

Numerical Experiments

Figure 2 displays first results of our algorithm when applied to exact, simulated data for solenoidal vector fields. The scanning curve was $\Gamma = rS^2 \cap \{x_3 = 0\}$, that is a circle of radius $r > 0$ in the plane $\{x_3 = 0\}$. The mollifier e_γ defining the fields $\mathbf{E}_\gamma^j$ was chosen as the Gaussian (7) and the reconstruction was done in the plane $\{x_3 = 0\}$. On the left the vector field $\mathbf{f}_1(x) = \left(1-x_2^2-x_3^2,0,0\right)^\top$ and its reconstruction are depicted, $\mathbf{f}_2(x) = (-x_2,x_1,0)^\top$ and the result of the described algorithm are shown on the right. The regularization parameter was $\gamma = 0.007$ and $\gamma = 0.00692$ respectively.



Top-Left: Original vector field $\mathbf{f}_1(x) = \left(1-x_2^2-x_3^2,0,0\right)^\top$.
Top-Right: Original vector field $\mathbf{f}_2(x) = (-x_2,x_1,0)^\top$.
Middle-Left: Reconstruction for $\gamma = 0.007$ using exact, simulated data $\mathbf{D}_1\mathbf{f}_1$.
Middle-Right: Reconstruction for $\gamma = 0.00692$ using exact, simulated data $\mathbf{D}_1\mathbf{f}_2$.
Right: Example of a reconstruction kernel for $\gamma = 0.007$.

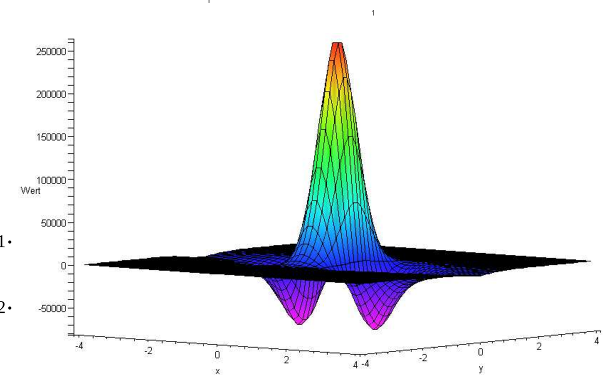


Figure 2

References

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