Harmonic Analysis Methods and the Regularity Problem for PDEs with Discontinuous Data

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CONTENTS

1. Introduction

The general aim of the present course is to show how results of the Real and the Harmonic Analysis apply to develop a relevant L^p-regularity and solvability theory of second-order linear elliptic partial differential equations (PDEs) with discontinuous coefficients. The presentation is as possible as self-contained, and more detailed and exhaustive exposition can be found in the monograph [9]. Basic knowledge of Functional Analysis and of second order elliptic PDEs with constant (eventually smooth) coefficients is only required.

In Section 2 we introduce the basic functional spaces to be used in the sequel. The Hardi–Littlewood and the sharp maximal operators are defined and their fundamental properties are summarized. In Section 3 we recall some known results regarding the Poisson equation and the regularity of the Newtonian potential in various functional spaces. Section 4 is the analytic heart of the course. We give here a detailed exposition of the boundedness properties in L^p of the singular integral operators of Calderón–Zygmund type and their commutators with the multiplication by a function of bounded mean oscillation. These results are applied in Section 5 to the L^p-regularity problem of second-order, linear elliptic equations with discontinuous principal coefficients of vanishing mean oscillation. We propose detailed study of the interior and boundary regularity of the strong solutions, deriving the corresponding L^p-estimates for the second derivatives. These estimates lead then to a global *a priori* estimate in the Sobolev space $W^{2,p}$ for any strong solution. Strong solvability of the Dirichlet problem for the operators considered is studied as well. Finally, in Section 6 we outline possible extensions of the results presented to another differential operators or problems.

Notations. We adopt in the sequel the following standard notations:

$\mathbb{N}$ = the set of positive integers, $\mathbb{Z}$ = the set of all integers, $\mathbb{R}$ = the set of real numbers, $\mathbb{R}_+$ = the set of real positive numbers

$\mathbb{R}^n$ = Euclidean n-dimensional space, $n \geq 2$, with points $x = (x_1, \dots, x_n)$, $|x| = \|x\|_{\mathbb{R}^n} = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$

$\omega_n = \dfrac{2\pi^{n/2}}{n\Gamma(n/2)}$ = the volume of the unit ball of $\mathbb{R}^n$

$\mathbb{S}^{n-1}$ = the unit sphere in $\mathbb{R}^n$, $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$

$B_\rho(x)$ = the Euclidean ball centered at the point x and of radius ρ, that is, $B_\rho(x) := \{y \in \mathbb{R}^n : \quad |y - x| < \rho\}$

$|E|$ = the Lebesgue measure of a set E

For a point set S define: ∂S = the boundary of S, $\overline{S}$ = the closure of $S = S \cup \partial S$; $S' \Subset S$ is S' has a compact closure in S

Ω = a domain in $\mathbb{R}^n$, that is, an open and connected subset of $\mathbb{R}^n$

For a function f with domain $\Omega \subseteq \mathbb{R}^n$ define $\operatorname{supp} f$ = support of f = the closure of Ω where $f \neq 0$

$C^k(\Omega)$ $(C^k(\overline{\Omega}))$ = the set of functions having continuous in Ω $(\overline{\Omega})$ derivatives up to order $k \in \mathbb{N} \cup \{0\}$, $k \le \infty$

$C_0^k(\Omega)$ = the set of functions $f \in C^k(\Omega)$ with $\operatorname{supp} f \Subset \Omega$

$D_i := \partial/\partial x_i$, $D_{ij} := \partial^2/\partial x_i \partial x_j$, $D^\alpha := D^{|\alpha|}/\partial^{\alpha_1} x_1 \dots \partial^{\alpha_n} x_n$, $\alpha = (\alpha_1, \dots, \alpha_n)$, $|\alpha| = \sum_{i=1}^n \alpha_i$

$Df = \big(D_1 f, \dots, D_n f\big)$ = the gradient of a function f

$D^2 f = \big\{D_{ij} f\big\}_{i,j=1}^n$ = the Hessian matrix of a function f

For a Lebesgue measurable set $E \in \mathbb{R}^n$ and an integrable function $f\colon E \to \mathbb{R}$ denote by f_E the integral average of f over E :

$$f_E := \frac{1}{|E|} \int_E f(x)\, dx = \fint_E f(x)\, dx$$

2. Function Spaces

2.1. Domains and their boundaries. Hereafter Ω will be a bounded domain (open and connected set) in $\mathbb{R}^n$ with $n \ge 2$. Sometimes we will require additionally some minimal smoothness of the boundary $\partial\Omega$. Namely, Ω will be said to support the (A_Ω)-*property* if there exists a constant $A_\Omega \in (0,1)$ such that

(A) $$A_\Omega \rho^n \le |B_\rho(x) \cap \Omega| \le (1 - A_\Omega)\rho^n \quad \forall x \in \partial\Omega,\ \forall \rho \in (0, \operatorname{diam} \Omega],$$

where $B_\rho(x)$ is the Euclidean ball centered at the point x and of radius ρ.

The lower bound in (A) excludes interior cusps at each point of $\partial\Omega$ and this guarantees the validity of the Sobolev embedding theorem in the spaces $W^{1,p}(\Omega)$. On the other hand, no exterior cusps exist on $\partial\Omega$ by means of the upper bound in (A). For instance, the (A_Ω)-property is verified if $\partial\Omega$ supports the uniform interior (that means there is a fixed cone K_Ω such that each $x \in \overline{\Omega}$ is a vertex of a cone $K_\Omega(x)$ congruent to K_Ω and such that $K_\Omega(x) \subset \overline{\Omega}$) and the exterior cone conditions. In particular, (2.1) holds in the cases of C^1-smooth, or Lipschitz continuous boundaries.

2.2. Lebesgue and Sobolev spaces. Given a number $p \in [1, \infty)$ and a measurable set $E \subseteq \mathbb{R}^n$, we will denote by $L^p(E)$ the Banach space of all measurable functions $f\colon E \to \mathbb{R}$ equipped with the norm

$$\|f\|_{L^p(E)} := \left(\int_E |f(x)|^p\, dx \right)^{1/p} < \infty,$$

while $L^\infty(E)$ stands for the space of the essentially bounded functions, with a norm given by

$$\|f\|_{L^\infty(E)} := \operatorname*{ess\,sup}_{x \in E} |f(x)|.$$

Further on, the local space $L^p_{\text{loc}}(E)$ is the collection of all measurable functions belonging to $L^p(E')$ for each $E' \Subset E \subseteq \mathbb{R}^n$.

Given an integer $k \in \mathbb{N}$ and a $p \geq 1$, we will denote by $W^{k,p}(E)$ the *Sobolev space* of k-times weakly differentiable functions $f\colon E \to \mathbb{R}$, belonging to $L^p(E)$ together with all derivatives $D^\alpha f$ for $|\alpha| =\leq k$, for which

$$\|f\|_{W^{k,p}(E)} := \left(\sum_{|\alpha|=0}^{k} \int_E |D^\alpha f(x)|^p \, dx \right)^{1/p} < \infty. \tag{2.1}$$

The closure of $C_0^k(E)$ with respect to the norm (2.1) will be denoted, as usual, by $W_0^{k,p}(E)$.

We will suppose hereafter that $\Omega \subset \mathbb{R}^n$ is a bounded domain.

Theorem 2.1 (**Sobolev embedding theorem**). *The space* $W_0^{1,p}(\Omega)$ *is* continuously embedded *into* $L^{np/(n-p)}(\Omega)$ *if* $p < n$, *and into* $C^0(\overline{\Omega})$ *if* $p > n$,

$$W_0^{1,p}(\Omega) \begin{cases} \nearrow L^{np/(n-p)}(\Omega) & \text{if } \ p < n, \\ \searrow C^0(\overline{\Omega}) & \text{if } \ p > n. \end{cases}$$

Moreover, there exists a constant $C = C(n,p)$ *such that*

$$\begin{aligned} \|f\|_{L^{np/(n-p)}(\Omega)} &\leq \|Df\|_{L^p(\Omega)} \quad \text{if } p < n, \\ \sup_\Omega |f| &\leq C|\Omega|^{1/n-1/p}\|Df\|_{L^p(\Omega)} \quad \text{if } p > n. \end{aligned}$$

More generally,

$$W_0^{k,p}(\Omega) \begin{cases} \nearrow L^{np/(n-kp)}(\Omega) & \text{if } \ kp < n, \\ \searrow C^m(\overline{\Omega}) & \text{for } \ 0 \leq m < k - n/p. \end{cases}$$

It should be noted that $W_0^{1,p}(\Omega)$ is continuously embedded into the *Orlicz space* $L^\varphi(\Omega)$ with a generic function $\varphi(t) = \exp\left(|t|^{n/(n-1)}\right) - 1$ when $p = n$, and into the *Hölder* space $C^{0,\alpha}(\overline{\Omega})$ with $\alpha = 1 - n/p$ when $p > n$ (see the Morrey Lemma 2.3 below). The claims of Theorem 2.1 remain valid if substitute $W_0^{k,p}(\Omega)$, $k \geq 1$, with $W^{k,p}(\Omega)$ for large class of domains Ω that satisfy (A).

Theorem 2.2 (**Poincaré inequality**). *Let* $f \in W_0^{1,p}(\Omega)$ *with* $p \geq 1$. *Then*

$$\|f\|_{L^p(\Omega)} \leq \left(\frac{|\Omega|}{\omega_n} \right)^{1/n} \|Df\|_{L^p(\Omega)},$$

where ω_n *is the volume of the unit ball of* $\mathbb{R}^n$.

If $f \in W^{1,p}(\Omega)$ *and* Ω *is a* convex domain, *then*

$$\left\| f - \fint_\Omega f(y)\, dy \right\|_{L^p(\Omega)} \leq \left(\frac{\omega_n}{|\Omega|} \right)^{1-1/n} (\operatorname{diam} \Omega)^n \|Df\|_{L^p(\Omega)}. \tag{2.2}$$

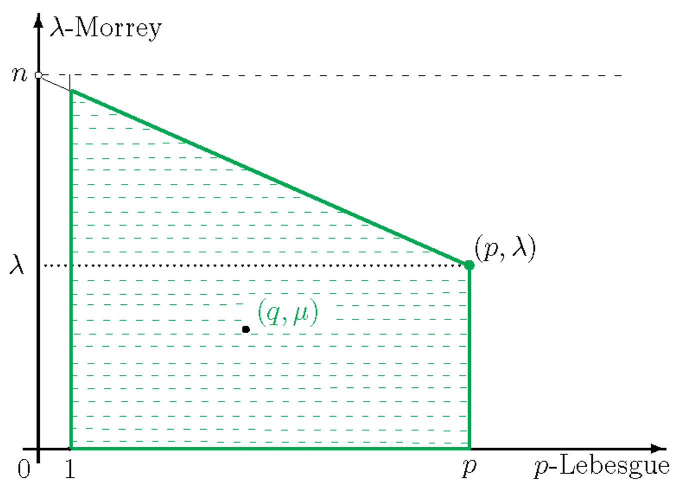
2.3. Morrey and Campanato spaces. Given a bounded domain Ω with the property (A) and two numbers $p \geq 1$ and $\lambda \in [0, n]$, we define the *Morrey space* $L^{p,\lambda}(\Omega)$ as the collection of all functions $f \in L^p(\Omega)$ such that

$$\|f\|_{L^{p,\lambda}(\Omega)} := \left(\sup_{x_0 \in \Omega,\ \rho > 0} \frac{1}{\rho^\lambda} \int_{B_\rho(x_0) \cap \Omega} |f(x)|^p \, dx \right)^{1/p} < \infty.$$

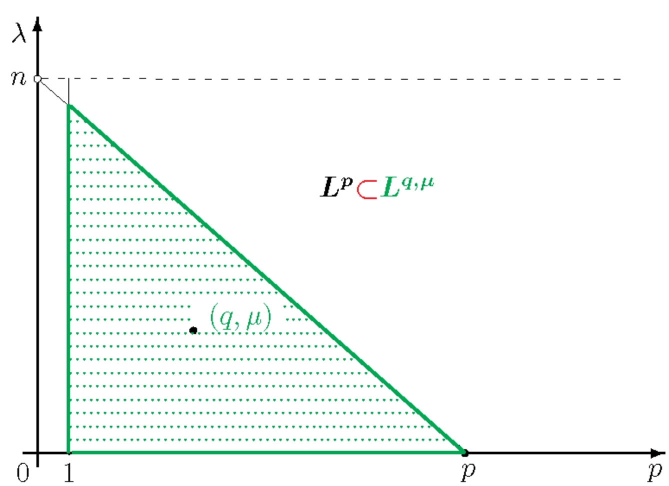
The space $L^{p,\lambda}(\Omega)$ equipped with the norm $\|\cdot\|_{L^{p,\lambda}(\Omega)}$ becomes a Banach space, and the limit cases $\lambda = 0$ and $\lambda = n$ give rise, respectively, to $L^p(\Omega)$ and $L^\infty(\Omega)$.

Inclusions:

1) $L^{p,\lambda} \subseteq L^{q,\mu} \iff \begin{cases} p \geq q, \\ \dfrac{p}{n-\lambda} \geq \dfrac{q}{n-\mu} \end{cases}$



2) $f \in L^p \Longrightarrow f \in L^{q,\mu}\ \forall q \leq p,\ \dfrac{n\mu}{n-\mu} \leq p$



3) $f \in L^{p,\lambda} \not\Longrightarrow \exists q > p\colon\ f \in L^q$

Lemma 2.3 (**Morrey Lemma**). *Let Ω be a bounded domain satisfying the property* (A) *and $f \in W^{1,p}(\Omega)$ such that $Df \in L^{p,\lambda}(\Omega)$ with $p + \lambda > n$.*

Then f is Hölder continuous in $\overline{\Omega}$, that is,

$$\sup_{x,y\in\overline{\Omega},\ x\neq y} \frac{|f(x)-f(y)|}{|x-y|^{\alpha}},$$

with exponent $\alpha = 1 - \dfrac{n-\lambda}{p}$.

Let $p \geq 1$ and $\lambda \in [0, n+p]$. The *Campanato space* $\mathcal{L}^{p,\lambda}(\Omega)$ is formed by these functions $f \in L^p(\Omega)$ for which

$$\|f\|_{\mathcal{L}^{p,\lambda}(\Omega)} := \left(\sup_{x_0\in\Omega,\ \rho>0} \frac{1}{\rho^{\lambda}} \int_{B_\rho(x_0)\cap\Omega} \left| f(x) - f_{B_\rho(x_0)\cap\Omega} \right|^p \, dx \right)^{1/p} < \infty,$$

with $f_{B_\rho(x_0)\cap\Omega}$ being the integral average of f over $B_\rho(x_0)\cap\Omega$.

Known facts:

1) $\mathcal{L}^{p,\lambda} \equiv L^{p,\lambda}$ if $\lambda \in (0,n)$;
2) $\mathcal{L}^{p,n} \equiv \mathcal{L}^{q,n}$ $\forall p,q \geq 1$ $\big(\equiv \mathcal{L}^{1,n} \equiv \mathrm{BMO}\big)$;
3) $\mathcal{L}^{p,\lambda} \equiv C^{0,\alpha}$ if $\lambda \in (n, n+p]$, $\alpha = \dfrac{\lambda - n}{p}$;

2.4. The John–Nirenberg space BMO of functions with bounded mean oscillation. A locally integrable function $f\colon \mathbb{R}^n \to \mathbb{R}$ is said to belong to BMO (see John–Nirenberg [8]) if

$$\|f\|_* := \sup_B ⨍_B |f(x) - f_B| \, dx < \infty$$

with sup taken over all balls in $\mathbb{R}^n$.
$f \in \mathrm{BMO}(\Omega)$ if $\chi_\Omega f \in \mathrm{BMO}$, where χ_Ω is the characteristic function of Ω.

Known facts:

1) $\|f\|_* = 0 \iff f = \text{const}$.
   Therefore, $\|\cdot\|_*$ defines a norm in BMO modulo the constant functions;
2) $L^\infty \subsetneqq \mathrm{BMO}$ and the inclusion is *proper*.
   For example, $\log|x| \in \mathrm{BMO}$ but $\log|x| \notin L^\infty$;
3) $f \in W^{1,1}$ with $Df \in L^{1,n-1} \Longrightarrow f \in \mathrm{BMO}$.
   In fact,

$$\frac{1}{|B_\rho|} \int_{B_\rho} \left| f - f_{B_\rho} \right| \, dx \leq \frac{C}{|B_\rho|} \left(\frac{1}{|B_\rho|} \right)^{1-1/n} \rho^n \int_{B_\rho} |Df| \, dx$$
$$\leq C \frac{1}{\rho^{n-1}} \int_{B_\rho} |Df| \, dx < \infty$$

in view of the Poincaré inequality (2.2);

4) $\mathcal{L}^{1,n} \equiv \mathrm{BMO}$;

5) **The John–Nirenberg Lemma:** *Let* $f \in \mathrm{BMO}$. *Then*

$$\left(\frac{1}{|B|} \int_B |f(x) - f_B|^p \, dx \right)^{1/p} \leq C(p) \|f\|_* \quad \forall p \geq 1. \tag{2.3}$$

6) $f \in \mathrm{BMO} \Longrightarrow f \in L^p \; \forall p \geq 1$;

7) $L^\infty \subset \mathrm{BMO} \subset L^p \; \forall p \geq 1$.

2.5. The Sarason class VMO of functions with vanishing mean oscillation. Let $f \in \mathrm{BMO}$ and set

$$\gamma_f(r) := \sup_{\rho \leq r} \frac{1}{|B_\rho|} \int_{B_\rho} \left| f(x) - f_{B_\rho} \right| \, dx,$$

where $B_\rho \in \mathbb{R}^n$ is any ball of radius ρ. The function f is said to belong to VMO (see Sarason [10]) if

$$\lim_{r \to 0^+} \gamma_f(r) = 0$$

and γ_f is referred as a VMO-modulus of f.
$f \in \mathrm{VMO}(\Omega)$ if $\chi_\Omega f \in \mathrm{VMO}$.

Known facts:

1) $\mathrm{BUC} \subset \mathrm{VMO}$ with γ_f being the modulus of continuity (BUC=bounded and uniformly continuous);

2) $W^{1,n} \subset \mathrm{VMO}$. In fact, the Poincaré inequality (2.2) yields

$$\frac{1}{|B|} \int_B |f(x) - f_B| \, dx \leq C(n) \underbrace{\left(\int_B |Df|^n \, dx \right)^{1/n}}_{\to 0 \text{ as } |B| \to 0}.$$

However, $W^{1,n} \subsetneqq \mathrm{VMO}$, that is, the inclusion is *proper*;
Examples: Define $f_\alpha(x) := \left| \log |x| \right|^\alpha$ with $\alpha \in (0,1)$. Then
- $f_\alpha \in \mathrm{VMO} \; \forall \alpha \in (0,1)$;
- $f_\alpha \in W^{1,n}$ for $\alpha \in \left(0, 1 - \frac{1}{n}\right)$;
- $f_\alpha \notin W^{1,n}$ for $\alpha \in \left[1 - \frac{1}{n}, 1\right)$;
- $f_1 = \log |x| \in \mathrm{BMO}$ but $\notin \mathrm{VMO}$.

3) **A useful criterion:** if $f(|x|)$ is a radially symmetric function, $f(r) \in C^1(0,R)$ and $\lim_{r \to 0^+}(r f'(r)) = 0$ then $f \in \mathrm{VMO}$;

4) $W^{\theta, n/\theta} \subset \mathrm{VMO} \; \forall \theta \in (0,1)$, where $W^{s,p}(\Omega)$, $s = [s] + \sigma$, is the *fractional Sobolev space* with norm

$$\|f\|_{W^{s,p}(\Omega)} := \left(\|f\|^p_{L^p(\Omega)} + \sum_{|\alpha| = [s]} \int_\Omega \int_\Omega \frac{\left| D^\alpha f(x) - D^\alpha f(y) \right|^p}{|x - y|^{n + \sigma p}} \, dx dy \right)^{1/p}.$$

In fact, the Hölder and the Jensen inequalities give

$$\begin{aligned}
\frac{1}{|B|}\int_B |f(x)-f_B|\,dx &\le \left(\frac{1}{|B|}\int_B \left|f(x)-\frac{1}{|B|}\int_B f(y)\,dy\right|^{n/\theta} dx\right)^{\theta/n}\\
&\le \left(\frac{1}{|B|}\int_B \left|\frac{1}{|B|}\int_B |f(x)-f(y)|\,dy\right|^{n/\theta} dx\right)^{\theta/n}\\
&\le \left(\frac{1}{|B|}\int_B \frac{1}{|B|}\int_B |f(x)-f(y)|^{n/\theta}\,dxdy\right)^{\theta/n}\\
&\le C(n)\left(\int_B\int_B \frac{|f(x)-f(y)|^{n/\theta}}{|x-y|^{2n}}\,dxdy\right)^{\theta/n} < \infty
\end{aligned}$$

and the last term tends to 0 as $|B| \to 0$ by the absolute continuity of the Lebesgue integral;

5) C^0 is a *proper* subspace of VMO : $\big|\log|x|\big|^{\alpha},\ \log\big|\log|x|\big| \in \mathrm{VMO}, \notin C^0$;

6) $f \in \mathrm{VMO}$ with $\gamma_f(r) \le cr^{\alpha} \implies f \in C^{0,\alpha}$:

$$\frac{1}{|B_r|}\int_{B_r} \big|f(x)-f_{B_r}\big|\,dx \le cr^{\alpha} \implies \frac{1}{r^{n+\alpha}}\int_{B_r}\big|f(x)-f_{B_r}\big|\,dx \le c$$
$$\implies f \in \mathcal{L}^{1,n+\alpha},\ n+\alpha > n \implies f \in C^{0,\alpha};$$

7) **Question:** *When, depending on* γ_f, *a* VMO *function is continuous?*
Answer: *The Spanne spaces* $\mathcal{L}_\Phi$*:* Let $\Phi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a non-decreasing function and $f \in L^1_{\mathrm{loc}}$. Then $f \in \mathcal{L}_\Phi$ if

$$\sup_B \frac{1}{\Phi(|B|^{1/n})}\frac{1}{|B|}\int_B \big|f(x)-f_B\big|\,dx =: \|f\|_{\mathcal{L}_\Phi} < \infty$$

with B ranging over the balls in $\mathbb{R}^n$.

- If $\Phi(t) = t^{\lambda-n}$ then $\mathcal{L}_\Phi \equiv \mathcal{L}^{1,\lambda}$;
- Let $f \in \mathcal{L}_\Phi$. Then, for each $B_\rho \subseteq B_r$:

$$\frac{1}{|B_\rho|}\int_{B_\rho}\big|f(x)-f_{B_\rho}\big|\,dx \le \|f\|_{\mathcal{L}_\Phi}\Phi(c(n)\rho) \le \|f\|_{\mathcal{L}_\Phi}\Phi(c(n)r).$$

  Therefore, $\mathcal{L}_\Phi \subseteq \mathrm{VMO}$ if $\lim_{r\to 0^+}\Phi(r) = 0$.
- If $\Phi \in C^{\mathrm{Dini}}$, that is, $\int_0 \frac{\Phi(t)}{t}\,dt < \infty$, then $\mathcal{L}_\Phi \subset C^0$ and this is *optimal* [13]

8) *Characterization of* VMO (Sarason [10]):
For $f \in \mathrm{BMO}$ the followings are equivalent:
   i) $f \in \mathrm{VMO}$;
   ii) f belongs to the closure of BUC with respect to BMO;
   iii) $\lim_{h\to 0}\big\|f(\cdot-h)-f(\cdot)\big\|_* = 0$.

In particular, ii) yields that, given $f \in \mathrm{VMO}$ with a modulus γ_f, there exists a sequence $\{f_k\} \in C^0(\mathbb{R}^n)$ such that

$$\lim_{k\to\infty}\|f_k - f\|_* = 0, \quad \gamma_{f_k} \le \gamma_f.$$

2.6. Hardi–Littlewood maximal operator. Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then

$$\mathcal{M}f(x) := \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)|\, dy \quad \text{for almost all } x \in \mathbb{R}^n.$$

It follows by the Lebesgue differentiation theorem that

$$|f(x)| \le \mathcal{M}f(x) \quad \text{for a.a. } x \in \mathbb{R}^n.$$

Maximal inequality ([14]): There exists a constant $C = C(n,p) > 0$ such that

$$\frac{1}{C}\|f\|_{L^p(\mathbb{R}^n)} \le \|\mathcal{M}f\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)} \quad \forall f \in L^p(\mathbb{R}^n),\ p \in (1,\infty]. \tag{2.4}$$

Moreover, $\mathcal{M}\colon\ \mathrm{BMO} \longrightarrow \mathrm{BMO}$ is *continuous.*

Weak maximal inequality $p = 1$:

$$\big|\{x \in \mathbb{R}^n\colon\ \mathcal{M}f(x) > \lambda\}\big| \le \frac{C(n)}{\lambda} \int_{\mathbb{R}^n} |f(y)|\, dy \quad f \in L^1(\mathbb{R}^n).$$

Particular cases of more general results hold in weighted Lebesgue spaces L^p_ω, where ω is a Muckenhoupt weight. Namely, a locally integrable function $\omega > 0$ is said to belong to the Muckenhoupt class $\mathcal{A}_p$, $p \in (1,\infty)$, if

$$\left(\fint_B \omega(x)\, dx\right)\left(\fint_B \omega(x)^{1/(1-p)}\, dx\right)^{p-1} < \infty.$$

The following result, known as *Muckenhoupt theorem* holds true:

$$\mathcal{M}\colon L^p_\omega \longrightarrow L^p_\omega \quad \text{is continuous} \iff \omega \in \mathcal{A}_p.$$

2.7. Sharp maximal operator. Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then

$$f^\#(x) := \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} \big|f(y) - f_{B_r(x)}\big|\, dy \quad \text{for almost all } x \in \mathbb{R}^n.$$

Facts:

- $f \in \mathrm{BMO} \iff f^\# \in L^\infty$, $\|f\|_* = \|f^\#\|_{L^\infty}$;
- $f^\#(x) \le 2\mathcal{M}f(x)$.

Fefferman–Stein inequality ([6]): There exists a constant $C = C(n,p) > 0$ such that

$$\|f\|_{L^p(\mathbb{R}^n)} \le C\|f^\#\|_{L^p(\mathbb{R}^n)} \quad \forall f \in L^p(\mathbb{R}^n),\ p \in [1,\infty). \tag{2.5}$$

3. Poisson Equation and the Newtonian Potential

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Fix $y \in \Omega$ and define the normalized *fundamental solution* of $\Delta u = 0$:

$$\Gamma_\Delta(x-y) = \Gamma_\Delta(|x-y|) := \begin{cases} \dfrac{1}{n(2-n)\omega_n}|x-y|^{2-n} & \text{if } n > 2, \\ \dfrac{1}{2\pi}\log|x-y| & \text{if } n = 2. \end{cases}$$

Straightforward calculations give

$$D_{x_i}\Gamma_\Delta(x-y) = \frac{1}{n\omega_n}\frac{x_i - y_i}{|x-y|^n}, \quad |D\Gamma_\Delta| \le \frac{1}{n\omega_n}|x-y|^{1-n},$$

$$D_{x_ix_j}\Gamma_\Delta(x-y) = \frac{1}{n\omega_n}\frac{|x-y|^2\delta_{ij} - n(x_i-y_i)(x_j-y_j)}{|x-y|^{n+2}}, \quad |D^2\Gamma_\Delta| \le \frac{1}{\omega_n}|x-y|^{-n},$$

$$\Delta_x\Gamma_\Delta(x-y) = 0 \quad \forall x \ne y.$$

For $f \in L^1(\Omega)$ define the *Newtonian potential*

$$\mathcal{N}f(x) := \int_\Omega \Gamma_\Delta(x-y)f(y)\,dy.$$

Known facts (see [7, Chapters 2 and 4] for instance):

- If $\partial\Omega$ is smooth then for each $u \in C^2(\overline{\Omega})$ there holds the *Green representation:*

$$u(x) = \int_\Omega \Gamma_\Delta(x-y)\Delta u(y)\,dy + \int_{\partial\Omega}\left(u(y)\frac{\partial\Gamma_\Delta}{\partial\nu}(x-y) - \Gamma_\Delta(x-y)\frac{\partial u}{\partial\nu}(y)\right)\,d\sigma_y,$$

  where ν is the unit outward normal to $\partial\Omega$, and the surface integral above is a *harmonic function;*
- If u has a compact support in $\mathbb{R}^n$ then

$$u(x) = \int_\Omega \Gamma_\Delta(x-y)\Delta u(y)\,dy = \mathcal{N}(\Delta u)(x).$$

Regularity of $\mathcal{N}f$:

- $f \in C_0^\infty(\Omega) \Longrightarrow \mathcal{N}f \in C^\infty(\mathbb{R}^n)$. In fact,

$$\mathcal{N}f(x) = \int_\Omega \Gamma_\Delta(x-y)f(y)\,dy = \int_{\mathbb{R}^n}\Gamma_\Delta(x-y)f(y)\,dy = \int_{\mathbb{R}^n}\Gamma_\Delta(z)f(x-z)\,dz \in C^\infty;$$

- $f \in L^\infty \cap L^1 \Longrightarrow \mathcal{N}f \in C^1(\mathbb{R}^n) : D_{x_i}(\mathcal{N}f)(x) = \int_\Omega D_{x_i}\Gamma_\Delta(x-y)f(y)\,dy;$

- $f \in C^0(\Omega) \not\Longrightarrow \mathcal{N}f \in C^2(\mathbb{R}^n)$.
  Example: Let $n = 3$, $\Omega = B_{1/2}(0)$ and
$$f(x,y,z) = \begin{cases} \dfrac{z^2}{(x^2+y^2+z^2)\log(x^2+y^2+z^2)} & \text{if } (x,y,z) \neq (0,0,0), \\ 0 & \text{if } (x,y,z) = (0,0,0). \end{cases}$$
  Then
$$\mathcal{N}f(x,y,z) = \iiint_{B(0)} \frac{f(\bar{x},\bar{y},\bar{z})\, d\bar{x}d\bar{y}d\bar{z}}{\sqrt{(x-\bar{x})^2+(y-\bar{y})^2+(z-\bar{z})^2}}$$
  does not have second derivatives at $(0,0,0)$.
- $f \in C^{\mathrm{Dini}}_{\mathrm{loc}} \cap L^\infty \Longrightarrow \mathcal{N}f \in C^2(\mathbb{R}^n)$;
- $f \in C^{0,\alpha}_{\mathrm{loc}} \cap L^\infty \Longrightarrow \mathcal{N}f \in C^2(\mathbb{R}^n)$:
  $\Delta(\mathcal{N}f) = f$ in Ω yields, by the Divergence Theorem applied to a smooth Ω_0, $\Omega \Subset \Omega_0$, that $\forall x \in \Omega$:
$$D_{x_ix_j}(\mathcal{N}f)(x) = \underbrace{\int_{\Omega_0} D_{ij}\Gamma_\Delta(x-y)\big(f(x)-f(y)\big)\, dy}_{<\infty \text{ because of } f\in C^{0,\alpha}} - f(x)\int_{\partial\Omega_0} D_i\Gamma_\Delta(x-y)\nu_j(y)\, d\sigma_y$$
  with $\nu = (\nu_1,\dots,\nu_n)$ being the unit outward normal to $\partial\Omega_0$;
- $f \in C^{0,\alpha}_{\mathrm{loc}}$ is *not necessary* to have $\mathcal{N}f \in C^2(\mathbb{R}^n)$.
  Example: Let $n = 3$, $\Omega = B_{1/2}(0)$ and
$$f(x,y,z) = \begin{cases} \dfrac{1}{\log(x^2+y^2+z^2)} & \text{if } (x,y,z) \neq (0,0,0), \\ 0 & \text{if } (x,y,z) = (0,0,0). \end{cases}$$
  Then $f \in C^0(\Omega)$ and $\mathcal{N}f \in C^2$;
- $f \in C^{0,\alpha}_{\mathrm{loc}} \cap L^\infty \Longrightarrow \mathcal{N}f \in C^{2,\alpha}_{\mathrm{loc}}(\mathbb{R}^n)$ **(Schauder** [11, 12]**)** (see [7, Chapters 4 and 6]);
- $f \in L^p(\Omega)$, $p \in (1,\infty) \Longrightarrow \mathcal{N}f \in W^{2,p}(\Omega)$ **(Calderón–Zygmund** [1, 2]**)** and $\Delta(\mathcal{N}f) = f$ a.e. in Ω (see [7, Chapter 9]);
  Calderón–Zygmund inequality:
$$\big\|D^2(\mathcal{N}f)\big\|_{L^p(\mathbb{R}^n)} \leq C(n,p)\|f\|_{L^p(\Omega)}$$
  In the particular case $p = 2$:
$$\int_{\mathbb{R}^n} \big|D^2(\mathcal{N}f)\big|^2\, dx = \int_\Omega |f(x)|^2\, dx;$$

- The above result *fails* when $p = 1$ or $p = \infty$.
  Example, $p = 1$: Let $n = 2$, $\Omega = B_{1/2}(0)$ and

$$f(x,y) = \begin{cases} \dfrac{1}{(x^2+y^2)\big|\log(x^2+y^2)\big|^{3/2}} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

  It is easy to check that $f \in L^1(\Omega)$ but

$$D_{xx}(\mathcal{N}f) = \frac{-x^2 + (y^2 - x^2)\log(x^2+y^2)}{(x^2+y^2)^2\big|\log(x^2+y^2)\big|^{3/2}} \notin L^1.$$

  Actually, $\mathcal{N}$ acts continuously from L^1 into the *Hardy* space $\mathcal{H}^1$;
  Example, $p = \infty$: Let $n = 2$, $\Omega = B_1(0)$ and

$$f(x,y) = \begin{cases} \dfrac{8xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

  It follows $f \in L^\infty(\Omega)$ but

$$D_{xy}(\mathcal{N}f) = \frac{2(x^4+y^4)}{(x^2+y^2)^2} + \log(x^2+y^2) \notin L^\infty.$$

  Actually, $\mathcal{N}$ acts continuously from BMO into BMO.

4. Calderón–Zygmund Singular Integral Operators and Commutators

4.1. Operators with constant kernel.

Definition 4.1 (Calderón–Zygmund kernel). A function $k\colon\ \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ is called a *Calderón–Zygmund kernel (CZ-kernel)* if

a) $k \in C^\infty(\mathbb{R}^n \setminus \{0\})$;
b) k is homogeneous of degree $-n$, that is, $k(tx) = t^{-n}k(x)$, $\forall t > 0$;
c) Cancellation property: $\displaystyle\int_{\mathbb{S}^{n-1}} k(x)\, d\sigma_x = 0$.

Note: $D^2\Gamma_\Delta(x)$ is a CZ-kernel!

Theorem 4.2 (Calderón–Zygmund, [1]). *Let k be a CZ-kernel and $\varepsilon > 0$.*

For $f \in L^p(\mathbb{R}^n)$ with $p \in (1,\infty)$ define

$$\mathcal{K}_\varepsilon f(x) := \int_{|x-y|>\varepsilon} k(x-y) f(y)\, dy.$$

Then, for each $f \in L^p(\mathbb{R}^n)$ there exists $\mathcal{K}f \in L^p(\mathbb{R}^n)$ such that

$$\lim_{\varepsilon\to 0} \big\|\mathcal{K}_\varepsilon f - \mathcal{K}f\big\|_{L^p(\mathbb{R}^n)} = 0.$$

Moreover, the operator $\mathcal{K}\colon L^p(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n)$ *is bounded,*

$$\big\|\mathcal{K}f\big\|_{L^p(\mathbb{R}^n)} \le C(n,p)\left(\int_{\mathbb{S}^{n-1}} k^2\,d\sigma_x\right)^{1/2}\|f\|_{L^p(\mathbb{R}^n)} \tag{4.1}$$

and $\lim_{\varepsilon\to 0}\mathcal{K}_\varepsilon f(x) = \mathcal{K}f(x)$ *for almost all* $x \in \mathbb{R}^n$.

Note: Similar result holds true within $C^{0,\alpha}$ instead of L^p.

The operator

$$\mathcal{K}f(x) := \text{P.V.}\int_{\mathbb{R}^n} k(x-y)f(y)\,dy = \text{P.V. } k * f(x)$$

is called a *Calderón–Zygmund singular integral operator.*

Let k be a CZ-kernel and take $\varphi \in \text{BMO}$. Given $f \in L^p(\mathbb{R}^n)$, define the *commutator* $\mathcal{C}[\varphi, f]$ of $\mathcal{K}$ with the multiplication by φ :

$$\begin{aligned}\mathcal{C}[\varphi,f](x) &:= \varphi(x)\mathcal{K}f(x) - \mathcal{K}(\varphi f)(x)\\ &= \text{P.V.}\int_{\mathbb{R}^n} k(x-y)\big[\varphi(x)-\varphi(y)\big]f(y)\,dy.\end{aligned}$$

Theorem 4.3 (**Coifman–Rochberg–Weiss,** [5])**.** *Given* $\varphi \in \text{BMO}$, *the commutator* $\mathcal{C}[\varphi, f]$ *is* well defined *for each* $f \in L^p(\mathbb{R}^n)$ *with* $p \in (1,\infty)$. *Moreover, the operator*

$$\mathcal{C}[\varphi,\cdot]\colon L^p(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n)$$

is bounded *and*

$$\big\|\mathcal{C}[\varphi,f]\big\|_{L^p(\mathbb{R}^n)} \le C\|\varphi\|_*\|f\|_{L^p(\mathbb{R}^n)}$$

with $C = C\left(n,p,\|k\|_{L^2(\mathbb{S}^{n-1})}\right)$.

Note that in the more restrictive case $\varphi \in L^\infty$ ($\subset$ BMO), the *existence* of $\mathcal{C}[\varphi, f]$ follows directly from Theorem 4.2. We will prove now a slightly weaker version of Theorem 4.3, assuming $\varphi \in L^\infty$.

Theorem 4.4. *Let k be a CZ-kernel and* $\varphi \in L^\infty(\mathbb{R}^n)$.

Then $\mathcal{C}[\varphi,\cdot]\colon L^p(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n)$ *is* bounded *for each* $p \in (1,\infty)$ *and*

$$\big\|\mathcal{C}[\varphi,f]\big\|_{L^p(\mathbb{R}^n)} \le C\|\varphi\|_*\|f\|_{L^p(\mathbb{R}^n)}$$

with $C = C\left(n,p,\|k\|_{L^2(\mathbb{S}^{n-1})}\right)$.

Proof. We start with the following result which asserts a fundamental property of the CZ-kernels.

Lemma 4.5 (**Pointwise Hörmander's condition**)**.** *Let k be a CZ-kernel.*

Then $\forall B_r(x_0) \subset B_{2r}(x_0)$, $x_0 \in \mathbb{R}^n$, *one has*

$$\big|k(x-y) - k(x_0-y)\big| \le C(n,k)\frac{|x-x_0|}{|x_0-y|^{n+1}} \tag{4.2}$$

whenever $x \in B_r(x_0)$ *and* $y \notin B_{2r}(x_0)$.

Proof. We note that

$$\left|D_{x_i}k(x)\right| \le \frac{C}{|x|^{n+1}} \quad \forall x \in \mathbb{R}^n \setminus \{0\}$$

with $C = \sup_{\mathbb{S}^{n-1}} |Dk|$. In fact, using the $(-n)$-homogeneity of k, and setting $\bar{x} := x/|x| \in \mathbb{S}^{n-1}$, we have

$$\begin{aligned}\lim_{h\to 0}\left|\frac{k(x+he_i)-k(x)}{h}\right| &= \lim_{h\to 0}\left|\frac{k\big(|x|\bar{x}+he_i\big)-k\big(|x|\bar{x}\big)}{h}\right| \\ &= |x|^{-n}\lim_{h\to 0}\left|\frac{k\left(\bar{x}+\frac{h}{|x|}e_i\right)-k(\bar{x})}{\frac{h}{|x|}}\right|\cdot\frac{1}{|x|} \le \frac{C}{|x|^{n+1}}\end{aligned}$$

with e_i being the i-th coordinate vector.

Fix now $x \in B_r(x_0)$ and $y \notin B_{2r}(x_0)$. It follows by the mean value theorem that

$$k(x-y) - k(x_0-y) = \sum_{i=1}^{n} D_{x_i}k(\xi)(x-x_0)_i$$

with $\xi = t(x-y) + (1-t)(x_0-y)$ for some $t \in (0,1)$.

We have $|x-y| \ge r$, $|x_0-y| \ge 2r$ and

$$\begin{aligned}|x_0-y| &\le |\xi| + \big|(x_0-y)-\xi\big| = |\xi| + \big|t(x_0-y)-t(x-y)\big| = |\xi| + t|x_0-x| \\ &< |\xi| + |x_0-x| < |\xi| + r.\end{aligned}$$

This way, $2r \le |x_0-y| < |\xi| + r$ yields $r < |\xi|$ whence

$$2|\xi| > |x_0-y|.$$

Therefore,

$$\begin{aligned}\big|k(x-y)-k(x_0-y)\big| &\le \sum_{i=1}^{n}\big|D_{x_i}k(\xi)\big|\,\big|(x-x_0)_i\big| \\ &\le n|x-x_0|\frac{C}{|\xi|^{n+1}} \le C\frac{|x-x_0|}{|x_0-y|^{n+1}}\end{aligned}$$

with a constant C, depending on n and $\sup_{\mathbb{S}^{n-1}}|Dk|$, and this completes the proof of Lemma 4.5. □

Note: The kernel CZ-kernel k satisfies also the weaker *Hörmander integral condition*

$$\int_{|y|\ge 4|x|}\big|k(x-y)-k(y)\big|\,dy \le C \quad \text{independently of } x.$$

Lemma 4.6. *Let $\varphi \in \mathrm{BMO}$. Then,*

$$\begin{aligned}|\varphi_{B_{2r}} - \varphi_{B_r}| &\le C(n)\|\varphi\|_*, \\ \big|\varphi_{B_{2^j r}} - \varphi_{B_r}\big| &\le C(n)j\|\varphi\|_*\end{aligned} \tag{4.3}$$

for any ball B_r and any $j \in \mathbb{N}$.

Proof. It is immediate that

$$
\begin{aligned}
|\varphi_{B_{2r}} - \varphi_{B_r}| &= \left| \frac{1}{|B_r|} \int_{B_r} \big(\varphi(x) - \varphi_{B_{2r}}\big)\, dx \right| \\
&\le \frac{C(n)}{|B_{2r}|} \int_{B_{2r}} \big|\varphi(x) - \varphi_{B_{2r}}\big|\, dx \le C(n)\|\varphi\|_*.
\end{aligned}
$$

To obtain (4.3), we employ the triangle inequality and the above estimate to get

$$
\begin{aligned}
\big|\varphi_{B_{2^j r}} - \varphi_{B_r}\big| \le\ & \big|\varphi_{B_{2^j r}} - \varphi_{B_{2^{j-1} r}}\big| + \big|\varphi_{B_{2^{j-1} r}} - \varphi_{B_{2^{j-2} r}}\big| \\
& + \cdots + |\varphi_{B_{4r}} - \varphi_{B_{2r}}| + |\varphi_{B_r} - \varphi_{B_r}| \\
\le\ & C(n) j \|\varphi\|_*.
\end{aligned}
$$

□

Turning back to the proof of Theorem 4.4, let us observe that it suffices to prove

$$
\big(\mathcal{C}[\varphi, f]\big)^{\#}(x) \le C(n,q,k)\|\varphi\|_* \left\{ \Big(\mathcal{M}\big(|\mathcal{K}f|^q\big)(x)\Big)^{1/q} + \Big(\mathcal{M}\big(|f|^q\big)(x)\Big)^{1/q} \right\} \tag{4.4}
$$

with $q > 1$ and for a.a. $x \in \mathbb{R}^n$.

In fact, if (4.4) holds true then for $q \in (1, p)$ it follows

$$
\begin{aligned}
\big\|\mathcal{C}[\varphi, f]\big\|_{L^p(\mathbb{R}^n)} &\overset{(2.5)}{\le} C \left\| \big(\mathcal{C}[\varphi, f]\big)^{\#} \right\|_{L^p(\mathbb{R}^n)} \\
&\overset{(4.4)}{\le} C\|\varphi\|_* \left\{ \left\| \big(\mathcal{M}\big(|\mathcal{K}f|^q\big)\big)^{1/q} \right\|_{L^p(\mathbb{R}^n)} + \left\| \big(\mathcal{M}\big(|f|^q\big)\big)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \right\} \\
&\le C\|\varphi\|_* \left\{ \big\|\mathcal{M}\big(|\mathcal{K}f|^q\big)\big\|^{1/q}_{L^{p/q}(\mathbb{R}^n)} + \big\|\mathcal{M}\big(|f|^q\big)\big\|^{1/q}_{L^{p/q}(\mathbb{R}^n)} \right\} \\
&\overset{(2.4)}{\le} C\|\varphi\|_* \left\{ \big\||\mathcal{K}f|^q\big\|^{1/q}_{L^{p/q}(\mathbb{R}^n)} + \big\||f|^q\big\|^{1/q}_{L^{p/q}(\mathbb{R}^n)} \right\} \\
&\le C\|\varphi\|_* \left\{ \big\|\mathcal{K}f\big\|_{L^p(\mathbb{R}^n)} + \big\|f\big\|_{L^p(\mathbb{R}^n)} \right\} \\
&\overset{(4.1)}{\le} C\|\varphi\|_* \big\|f\big\|_{L^p(\mathbb{R}^n)}
\end{aligned}
$$

and we are done.

To get the estimate (4.4), we take $x_0 \in \mathbb{R}^n$, $x \in B_r(x_0)$ and denote $2^j B := B_{2^j r}(x_0)$ for the sake of simplicity. Decompose the commutator into

$$
\begin{aligned}
\mathcal{C}[\varphi, f](x) &= \varphi(x)\mathcal{K}f(x) - \mathcal{K}(\varphi f)(x) \\
&= \underbrace{\big(\varphi(x) - \varphi_B\big)\mathcal{K}f(x)}_{T_1(x)} - \underbrace{\mathcal{K}\Big(\big(\varphi - \varphi_B\big) f \chi_{2B}\Big)(x)}_{T_2(x)} \\
&\quad - \underbrace{\mathcal{K}\Big(\big(\varphi - \varphi_B\big) f \chi_{(2B)^c}\Big)(x)}_{T_3(x)},
\end{aligned}
$$

where χ_{2B} is the characteristic function of $2B$ and $\chi_{(2B)^c}$ is that of the complement $(2B)^c$.

Estimate of the term T_1: We use the Hölder and the John–Nirenberg (cf. (2.3)) inequalities:

$$\begin{aligned}\fint_B \big|T_1(x)-(T_1)_B\big|\,dx &\le 2\fint_B \big|T_1(x)\big|\,dx \le 2\fint_B \big|\big(\varphi(x)-\varphi_B\big)\mathcal{K}f(x)\big|\,dx\\ &\le 2\left(\fint_B \big|\varphi(x)-\varphi_B\big|^{q'}\,dx\right)^{1/q'}\left(\fint_B \big|\mathcal{K}f(x)\big|^{q}\,dx\right)^{1/q}\\ &\le C(q)\|\varphi\|_*\Big(\mathcal{M}\big(|\mathcal{K}f|^q\big)(x_0)\Big)^{1/q}\end{aligned}$$

with $1/q+1/q'=1$.

Estimate of the term T_2: Let $q>1$ and $s\in(1,q)$. Then

$$\begin{aligned}\fint_B \big|T_2(x)-(T_2)_B\big|\,dx \le\;& 2\fint_B \big|T_2(x)\big|\,dx \le 2\fint_B \Big|\mathcal{K}\big((\varphi-\varphi_B)f\chi_{2B}\big)(x)\Big|\,dx\\ \overset{\text{Hölder}}{\le}\;& 2\left(\fint_B \Big|\mathcal{K}\big((\varphi-\varphi_B)f\chi_{2B}\big)(x)\Big|^{s}\,dx\right)^{1/s}\\ \le\;& 2\left(\frac{1}{|B|}\int_{\mathbb{R}^n} \Big|\mathcal{K}\big((\varphi-\varphi_B)f\chi_{2B}\big)(x)\Big|^{s}\,dx\right)^{1/s}\\ \overset{(4.1)}{\le}\;& C(s,k)\left(\frac{1}{|B|}\int_{2B} \big|\varphi(x)-\varphi_B\big|^{s}\big|f(x)\big|^{s}\,dx\right)^{1/s}\\ \overset{\text{Hölder}}{\le}\;& C(s,k)\left\{\frac{1}{|B|}\left(\int_{2B}|\varphi-\varphi_B|^{\frac{qs}{q-s}}\right)^{\frac{q-s}{q}}\left(\int_{2B}|f|^q\right)^{\frac{s}{q}}\right\}^{1/s}.\end{aligned}$$

The first term on the right-hand side above will be managed with the aid of the John-Nirenberg inequality (2.3) and Lemma 4.6:

$$\begin{aligned}\int_{2B}\big|\varphi(x)-\varphi_B\big|^{\frac{qs}{q-s}}\,dx &\le C\int_{2B}\big|\varphi(x)-\varphi_{2B}+\varphi_{2B}-\varphi_B\big|^{\frac{qs}{q-s}}\,dx\\ &\le C\left(\int_{2B}\big|\varphi(x)-\varphi_{2B}\big|^{\frac{qs}{q-s}}\,dx+|2B|\big|\varphi_{2B}-\varphi_B\big|^{\frac{qs}{q-s}}\right)\\ &\le C\left(\|\varphi\|_*^{\frac{qs}{q-s}}|2B|+|2B|\|\varphi\|_*^{\frac{qs}{q-s}}\right).\end{aligned}$$

This way,

$$\begin{aligned}\fint_B \big|T_2(x)-(T_2)_B\big|\,dx &\le C(s,k)\left\{\frac{1}{|B|}\left(\int_{2B}|f|^q\right)^{\frac{s}{q}}|2B|^{\frac{q-s}{s}}\|\varphi\|_*^{s}\right\}^{1/s}\\ &\le C(q,k)\|\varphi\|_*\left\{\frac{1}{|B|^{s/q}}\left(\int_{2B}|f|^q\right)^{s/q}\right\}^{1/s}\\ &= C(n,q,k)\|\varphi\|_*\left(\frac{1}{|2B|}\int_{2B}\big|f(x)\big|^q\,dx\right)^{1/q}\\ &\le C(n,q,k)\|\varphi\|_*\Big(\mathcal{M}\big(|f|^q\big)(x_0)\Big)^{1/q}.\end{aligned}$$

Estimate of the term T_3: We note, first of all, that

$$⨍_B \big|T_3(x) - (T_3)_B\big|\,dx \le \frac{2}{|B|}\int_B \big|T_3(x) - T_3(x_0)\big|\,dx,$$

and the integrand will be estimated by means of the pointwise Hörmander condition. Namely,

$$\begin{aligned}
\big|T_3(x) - T_3(x_0)\big| &= \Big|\mathcal{K}\big((\varphi-\varphi_B)f\chi_{(2B)^c}\big)(x) - \mathcal{K}\big((\varphi-\varphi_B)f\chi_{(2B)^c}\big)(x_0)\Big| \\
&\le \int_{(2B)^c} \big|k(x-y)-k(x_0-y)\big|\,\big|\varphi(y)-\varphi_B\big|\,\big|f(y)\big|\,dy \\
&\overset{\text{Lemma 4.5}}{\le} C(n,k)\int_{(2B)^c}\frac{|x-x_0|}{|x_0-y|^{n+1}}\big|\varphi(y)-\varphi_B\big|\,\big|f(y)\big|\,dy \\
&\le C(n,k)r\left(\int_{(2B)^c}\frac{|f(y)|^q}{|x_0-y|^{n+1}}\,dy\right)^{1/q}\left(\int_{(2B)^c}\frac{\big|\varphi(y)-\varphi_B\big|^{q'}}{|x_0-y|^{n+1}}\,dy\right)^{1/q'}.
\end{aligned}$$

To estimate the first integral above we use dyadic decomposition of $(2B)^c$ to obtain

$$\begin{aligned}
\int_{(2B)^c}\frac{|f(y)|^q}{|x_0-y|^{n+1}}\,dy &= \sum_{j=2}^{\infty}\int_{2^jB\setminus 2^{j-1}B}\frac{|f(y)|^q}{|x_0-y|^{n+1}}\,dy \\
&\le \sum_{j=2}^{\infty}\frac{C}{2^jr}\frac{1}{|2^jB|}\int_{2^jB}\big|f(y)\big|^q\,dy \\
&\le \mathcal{M}\big(|f|^q\big)(x_0)\frac{C}{r}\sum_{j=2}^{\infty}\frac{1}{2^j}.
\end{aligned}$$

Similarly, employing the John–Nirenberg inequality (2.3) and Lemma 4.6, we obtain

$$\int_{(2B)^c}\frac{\big|\varphi(y)-\varphi_B\big|^{q'}}{|x_0-y|^{n+1}}\,dy \le \frac{C(n,q)}{r}\|\varphi\|_*^{q'}.$$

Therefore

$$\big|T_3(x) - T_3(x_0)\big| \le C(n,q,k)\|\varphi\|_*\Big(\mathcal{M}\big(|f|^q\big)(x_0)\Big)^{1/q}$$

and also

$$\begin{aligned}
⨍_B \big|T_3(x) - (T_3)_B\big|\,dx &\le \frac{2}{|B|}\int_B \big|T_3(x) - T_3(x_0)\big|\,dx \\
&\le C(n,q,k)\|\varphi\|_*\Big(\mathcal{M}\big(|f|^q\big)(x_0)\Big)^{1/q}.
\end{aligned}$$

The bounds obtained for the sharp functions of T_1, T_2 and T_3 yield the desired estimate (4.4) and this completes the proof of Theorem 4.4. □

4.2. Spherical harmonics. Let $P\colon \mathbb{R}^n \to \mathbb{R}$ be a homogeneous polynomial of degree $m \in \mathbb{N} \cup \{0\}$, which is a harmonic function, $\Delta P = 0$. Its restriction $P(x)\Big|_{\mathbb{S}^{n-1}}$ on the unit sphere $\mathbb{S}^{n-1}$ is called an *n-dimensional spherical harmonic of degree m.*

Set $\mathcal{H}_m$ for the linear space of all n-dimensional spherical harmonics of degree m. Indeed, $\mathcal{H}_0 \equiv \mathbb{R}$, $\mathcal{H}_1 = \left\{\sum_{i=1}^n c_i x_i\colon\ c_i \in \mathbb{R}\right\}, \ldots$.

Setting $g_m := \dim \mathcal{H}_m$, we will list some useful facts about the spherical harmonics.

The first one regards a bound for the dimensions g_m :

$$g_m = \binom{m+n-1}{n-1} - \binom{m+n-3}{n-1} \leq C(n) m^{n-2} \tag{4.5}$$

with the second binomial coefficient settled equal to 0 if $m = 0, 1$.
In particular, $g_0 = 1$, $g_1 = n$, and $g_m = 2\ \forall m \geq 1$ if $n = 2$.

Let $\left\{Y_{sm}(x)\right\}_{s=1}^{g_m}$ be an *orthonormal* base of $\mathcal{H}_m$. Then $\left\{Y_{sm}(x)\right\}_{s=1,\, m=0}^{g_m \quad \infty}$ is a *complete* orthonormal system in $L^2(\mathbb{S}^{n-1})$ and

$$\sup_{x \in \mathbb{S}^{n-1}} \left|D^\beta Y_{sm}(x)\right| \leq C(n) m^{|\beta| + \frac{n-2}{2}} \quad m = 1, 2, \ldots. \tag{4.6}$$

Let $\Phi \in C^\infty(\mathbb{S}^{n-1})$ and

$$\sum_{s,m} a_{sm} Y_{sm}(x) \qquad \left(\sum_{s,m} := \sum_{m=0}^{\infty} \sum_{s=1}^{g_m}\right)$$

be the Fourier series expansion of Φ,

$$a_{sm} = \int_{\mathbb{S}^{n-1}} \Phi(y) Y_{sm}(y)\, d\sigma_y.$$

Then

$$|a_{sm}| \leq C(n, \ell) m^{-2\ell} \sup_{\substack{|\beta| = 2\ell \\ y \in \mathbb{S}^{n-1}}} \left|D^\beta \Phi(y)\right| \quad \forall \ell \in \mathbb{N} \tag{4.7}$$

and the expansion in spherical harmonics converges *uniformly* to Φ.
Note: The proofs of (4.5), (4.6) and (4.7) can be found in [2] (see also [14]) and these make use of the operator $\Lambda u := |x|^2 \Delta u$ for which the following integration-by-parts formula holds on $\mathbb{S}^{n-1}$: if $f, g \in C^{2\ell}(\mathbb{R}^n \setminus \{0\})$ are homogeneous of degree 0 then

$$\int_{\mathbb{S}^{n-1}} f \Lambda^\ell g\, d\sigma = \int_{\mathbb{S}^{n-1}} g \Lambda^\ell f\, d\sigma.$$

So, it turns out that

$$Y_{sm} = (-m)^{-\ell} (m+n-2)^{-\ell} \Lambda^\ell Y_{sm} \quad \forall s = 1, \ldots, g_m,\ \forall \ell \in \mathbb{N}.$$

Example: Let $n = 2$ and denote the points in $\mathbb{R}^2$ by (x, y). It follows from (4.5) that $g_0 = 1$ and $g_m = 2\ \forall m \geq 1$.

Further on,

$\mathcal{H}_0 \equiv \mathbb{R}$, $g_0 = 1$,
$\{Y_{s0}\}_{s=1}^{1} = \{Y_{10} = 1\}$

$\mathcal{H}_1 = \{c_1 x + c_2 y\}$, $g_1 = 2$,
$\{Y_{s1}\}_{s=1}^{2} = \{Y_{11} = x,\ Y_{21} = y\} \equiv \{Y_{11} = \boldsymbol{\cos\varphi},\ Y_{21} = \boldsymbol{\sin\varphi}\}$
with $\varphi \in [0, 2\pi)$ since $(x, y) \in \mathbb{S}^1$

$\mathcal{H}_2 = \{\cdots\}$, $g_2 = 2$,

$$\begin{aligned}\{Y_{s2}\}_{s=1}^{2} = \{Y_{12} = x^2 - y^2,\ Y_{22} = xy\} &\equiv \{Y_{12} = \cos^2\varphi - \sin^2\varphi,\ Y_{22} = \sin\varphi\cos\varphi\} \\ &\equiv \{Y_{12} = \boldsymbol{\cos 2\varphi},\ Y_{22} = \boldsymbol{\sin 2\varphi}\}\end{aligned}$$

. . . .

This way, the standard trigonometric system appears.

4.3. Calderón–Zygmund operators and commutators with variable kernels. To apply the results of Theorems 4.2 and 4.3 to second-order elliptic equations, some their generalizations are needed which deal with singular integrals and commutators, depending on a parameter such as

$$\mathcal{K}f(x) = \text{P.V.} \int_{\mathbb{R}^n} k(x; x - y) f(y)\, dy$$

with *lack of regularity* in the kernel k with respect to the first variable.

We need thus the concept of *variable* Calderón–Zygmund kernel.

Definition 4.7 (variable Calderón–Zygmund kernel). A function

$$k\colon\ \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$$

is called a *variable Calderón–Zygmund kernel* if

a) $k(x; \cdot)$ is a *constant* CZ-kernel in the sense of Definition 4.1 for almost all fixed $x \in \mathbb{R}^n$;
b) $\sup_{y \in \mathbb{S}^{n-1}} \left|D_y^\beta k(x; y)\right| \le C(\beta)$ *independently* of x.

We are in a position now to extend the results of Theorems 4.2 and 4.3 to operators with *variable* kernels.

Theorem 4.8 (Chiarenza–Frasca–Longo, [3]**).** *Let $k(x; y)$ be a* variable *CZ-kernel and for $f \in L^p(\mathbb{R}^n)$, $p \in (1, \infty)$, $\varphi \in L^\infty(\mathbb{R}^n)$ and $\varepsilon > 0$ define*

$$\begin{aligned}\mathcal{K}_\varepsilon f(x) &:= \int_{|x-y|>\varepsilon} k(x; x - y) f(y)\, dy, \\ \mathcal{C}_\varepsilon[\varphi, f](x) &:= \varphi(x)\mathcal{K}_\varepsilon f(x) - \mathcal{K}_\varepsilon(\varphi f)(x) \\ &= \int_{|x-y|>\varepsilon} k(x; x - y)\big[\varphi(x) - \varphi(y)\big] f(y)\, dy.\end{aligned}$$

Then, for each $f \in L^p(\mathbb{R}^n)$ with $p \in (1, \infty)$ there exist $\mathcal{K}f, \mathcal{C}[\varphi, f] \in L^p(\mathbb{R}^n)$ such that

$$\lim_{\varepsilon \to 0} \left\|\mathcal{K}_\varepsilon f - \mathcal{K}f\right\|_{L^p(\mathbb{R}^n)} = \lim_{\varepsilon \to 0} \left\|\mathcal{C}_\varepsilon[\varphi, f] - \mathcal{C}[\varphi, f]\right\|_{L^p(\mathbb{R}^n)} = 0.$$

Moreover, there exists a constant $C = C\left(n, p, \max_{|\beta|=2n} \left\|D_y^\beta k(x;y)\right\|_{L^\infty(\mathbb{R}^n\times\mathbb{S}^{n-1})}\right)$, *such that*

$$\left\|\mathcal{K}f\right\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}, \tag{4.8}$$

$$\left\|\mathcal{C}[\varphi, f]\right\|_{L^p(\mathbb{R}^n)} \leq C\|\varphi\|_*\|f\|_{L^p(\mathbb{R}^n)}. \tag{4.9}$$

Proof. By density arguments, it will be sufficient to prove Theorem 4.8 assuming $f \in C_0^\infty(\mathbb{R}^n)$.

Consider the function

$$z \mapsto |z|^n k(x;z).$$

Thanks to the fact that k is a variable CZ-kernel, it lies in $C^\infty(\mathbb{R}^n \setminus \{0\})$ for a.a. $x \in \mathbb{R}^n$, it is homogeneous of degree 0 and satisfies the cancellation property

$$\int_{\mathbb{S}^{n-1}} |z|^n k(x;z)\, d\sigma_z = 0.$$

Let $\left\{Y_{sm}(x)\right\}_{s=1,\, m=0}^{g_m\quad \infty}$ be as before a complete orthonormal system in $L^2(\mathbb{S}^{n-1})$ of spherical harmonics and define

$$a_{sm}(x) = \int_{\mathbb{S}^{n-1}} k(x;\bar{z})Y_{sm}(\bar{z})\, d\sigma_{\bar{z}} \quad \forall m \in \mathbb{N},\ s = 1, \dots, g_m.$$

(Note that $a_{s0} = 0$ by the cancellation property of $k(x;y)$.)

It follows, by the completeness of $\left\{Y_{sm}(x)\right\}$ in $L^2(\mathbb{S}^{n-1})$, that

$$k(x;\bar{z}) = \sum_{m=1}^{\infty}\sum_{s=1}^{g_m} a_{sm}(x)Y_{sm}(\bar{z}) \quad \forall \bar{z} = \frac{z}{|z|} \in \mathbb{S}^{n-1},$$

whence

$$k(x;z) = \sum_{m=1}^{\infty}\sum_{s=1}^{g_m} a_{sm}(x)\frac{Y_{sm}(\bar{z})}{|z|^n} \quad \forall z \in \mathbb{R}^n.$$

Step 1: Bound for $\|a_{sm}\|_{L^\infty(\mathbb{R}^n)}$. It follows from (4.7) that

$$\left|a_{sm}(x)\right| \leq CMm^{-2n} \qquad \text{with } M := \max_{\substack{|\beta|=2n \\ y\in\mathbb{S}^{n-1}}} \left|D_y^\beta k(x;y)\right|. \tag{4.10}$$

Step 2: Series expansions for $\mathcal{K}_\varepsilon f$ and $\mathcal{C}_\varepsilon f[\varphi, f]$. Substituting the expansion of $k(x;z)$ into the definition of $\mathcal{K}_\varepsilon f$ yields

$$\mathcal{K}_\varepsilon f(x) = \int_{|x-y|>\varepsilon} \sum_{m=1}^{\infty}\sum_{s=1}^{g_m} a_{sm}(x)\frac{Y_{sm}(\overline{x-y})}{|x-y|^n} f(y)\, dy.$$

Now, for a.a. $x \in \mathbb{R}^n$ and all $y \in \mathbb{R}^n$ such that $|x-y| > \varepsilon$, the bounds (4.5), (4.6) and (4.10) give

$$\begin{aligned}\left|\sum_{m=1}^{\infty}\sum_{s=1}^{g_m} a_{sm}(x)\frac{Y_{sm}(\overline{x-y})}{|x-y|^n}f(y)\right| &\leq \frac{1}{\varepsilon^n}\big|f(y)\big|\sum_{m=1}^{\infty}\sum_{s=1}^{g_m}\|a_{sm}\|_{L^\infty(\mathbb{R}^n)}\|Y_{sm}\|_{L^\infty(\mathbb{S}^{n-1})}\\ &\leq \frac{C(n)}{\varepsilon^n}\big|f(y)\big|\sum_{m=1}^{\infty} m^{-2n}m^{\frac{n-2}{2}}m^{n-2}\\ &= \frac{C(n)}{\varepsilon^n}\big|f(y)\big|\sum_{m=1}^{\infty} m^{-\frac{n+6}{2}}\end{aligned}$$

with a convergent numerical series.

Thus, the Dominated Convergence Theorem yields

$$\mathcal{K}_\varepsilon f(x) = \sum_{m=1}^{\infty}\sum_{s=1}^{g_m} a_{sm}(x)\int_{|x-y|>\varepsilon}\frac{Y_{sm}(\overline{x-y})}{|x-y|^n}f(y)\,dy, \tag{4.11}$$

and in the same manner

$$\mathcal{C}_\varepsilon[\varphi,f](x) = \sum_{m=1}^{\infty}\sum_{s=1}^{g_m} a_{sm}(x)\int_{|x-y|>\varepsilon}\frac{Y_{sm}(\overline{x-y})}{|x-y|^n}\big[\varphi(x)-\varphi(y)\big]f(y)\,dy. \tag{4.12}$$

Step 3: Total convergence in $L^p(\mathbb{R}^n)$ of the series (4.11) and (4.12), uniformly with respect to ε.

Note that

$$\bar{k}(z) := \frac{Y_{sm}(\bar{z})}{|z|^n}, \quad z \in \mathbb{R}^n$$

is a *constant* CZ-kernel. In fact,

- $\bar{k} \in C^\infty(\mathbb{R}^n \setminus \{0\})$;
- $\bar{k}(tz) = \frac{Y_{sm}(\overline{tz})}{|tz|^n} = t^{-n}\frac{Y_{sm}(\bar{z})}{|z|^n} = t^{-n}\bar{k}(z)$, $t > 0$;
- $\int_{\mathbb{S}^{n-1}} \bar{k}(z)\,d\sigma_z = \int_{\mathbb{S}^{n-1}} \frac{Y_{sm}(\bar{z})}{|z|^n}\,d\sigma_z = \int_{\mathbb{S}^{n-1}} Y_{sm}(\bar{z})\,d\sigma_{\bar{z}} = 0$

since Y_{sm} is harmonic, homogeneous, $Y_{sm}(0) = 0$ and because of the mean value property on spheres of the harmonic functions.

Moreover, $\|\bar{k}\|_{L^2(\mathbb{S}^{n-1})} = 1$ by the orthonormality of Y_{sm}.

Define now

$$\begin{aligned}\mathcal{R}_{sm,\varepsilon}f(x) &:= \int_{|x-y|>\varepsilon}\frac{Y_{sm}(\overline{x-y})}{|x-y|^n}f(y)\,dy,\\ \mathcal{S}_{sm,\varepsilon}[\varphi,f](x) &:= \int_{|x-y|>\varepsilon}\frac{Y_{sm}(\overline{x-y})}{|x-y|^n}\big[\varphi(x)-\varphi(y)\big]f(y)\,dy.\end{aligned}$$

Theorems 4.2 and 4.4/4.3 ensure existence of *bounded* linear operators $\mathcal{R}_{sm}$ and $\mathcal{S}_{sm}$, acting from $L^p(\mathbb{R}^n)$ into itself, such that

$$\begin{aligned}\mathcal{R}_{sm}f &:= \lim_{\varepsilon\to 0}{}^{L^p(\mathbb{R}^n)}\mathcal{R}_{sm,\varepsilon}f,\\ \mathcal{S}_{sm}[\varphi,f] &= \lim_{\varepsilon\to 0}{}^{L^p(\mathbb{R}^n)}\mathcal{S}_{sm,\varepsilon}[\varphi,f].\end{aligned}$$

Moreover,

$$\begin{aligned}\big\|\mathcal{R}_{sm,\varepsilon}f\big\|_{L^p(\mathbb{R}^n)} &\leq C(n,p)\|f\|_{L^p(\mathbb{R}^n)},\\ \big\|\mathcal{S}_{sm,\varepsilon}[\varphi,f]\big\|_{L^p(\mathbb{R}^n)} &\leq C(n,p)\|\varphi\|_*\|f\|_{L^p(\mathbb{R}^n)}.\end{aligned}$$

Using once again (4.10) and (4.5), we have

$$\sum_{m=1}^{\infty}\sum_{s=1}^{g_m}\big\|a_{sm}\mathcal{R}_{sm,\varepsilon}f\big\|_{L^p(\mathbb{R}^n)} \leq C(n,p,M)\|f\|_{L^p(\mathbb{R}^n)}\sum_{m=1}^{\infty}m^{-2n}m^{n-2}$$

and similarly

$$\sum_{m=1}^{\infty}\sum_{s=1}^{g_m}\big\|a_{sm}\mathcal{S}_{sm,\varepsilon}[\varphi,f]\big\|_{L^p(\mathbb{R}^n)} \leq C(n,p,M)\|f\|_{L^p(\mathbb{R}^n)}\|\varphi\|_*\sum_{m=1}^{\infty}m^{-2n}m^{n-2}$$

for (4.12)

Step 4: Series expansions for $\mathcal{K}f$ and $\mathcal{C}[\varphi,f]$.

Define

$$\begin{aligned}\mathcal{K}f(x) &:= \sum_{m=1}^{\infty}\sum_{s=1}^{g_m}a_{sm}(x)\mathcal{R}_{sm}f(x),\\ \mathcal{C}[\varphi,f](x) &:= \sum_{m=1}^{\infty}\sum_{s=1}^{g_m}a_{sm}(x)\mathcal{S}_{sm}[\varphi,f](x).\end{aligned}$$

It follows from Theorems 4.2 and 4.4/4.3 that

$$\begin{aligned}\big\|\mathcal{R}_{sm}f\big\|_{L^p(\mathbb{R}^n)} &\leq C(n,p)\|f\|_{L^p(\mathbb{R}^n)},\\ \big\|\mathcal{S}_{sm}[\varphi,f]\big\|_{L^p(\mathbb{R}^n)} &\leq C(n,p)\|\varphi\|_*\|f\|_{L^p(\mathbb{R}^n)}\end{aligned}$$

and, as above,

$$\begin{aligned}\big\|\mathcal{K}f\big\|_{L^p(\mathbb{R}^n)} &\leq C(n,p,M)\|f\|_{L^p(\mathbb{R}^n)}\sum_{m=1}^{\infty}m^{-2n}m^{n-2},\\ \big\|\mathcal{C}[\varphi,f]\big\|_{L^p(\mathbb{R}^n)} &\leq C(n,p,M)\|f\|_{L^p(\mathbb{R}^n)}\|\varphi\|_*\sum_{m=1}^{\infty}m^{-2n}m^{n-2}\end{aligned}$$

and these last prove (4.8) and (4.9).

To complete the proof of Theorem 4.8, it remains to show

$$\lim_{\varepsilon\to 0}\big\|\mathcal{K}_\varepsilon f-\mathcal{K}f\big\|_{L^p(\mathbb{R}^n)} = \lim_{\varepsilon\to 0}\big\|\mathcal{C}_\varepsilon[\varphi,f]-\mathcal{C}[\varphi,f]\big\|_{L^p(\mathbb{R}^n)} = 0.$$

For, pass to the limit in (4.11) and (4.12) as $\varepsilon \to 0$. By the Step 3 it is possible to commute the operations of passing to the limit with the summation, whence

$$\lim_{\varepsilon\to 0}{}^{L^p(\mathbb{R}^n)}\mathcal{K}_\varepsilon f = \sum_{m=1}^{\infty}\sum_{s=1}^{g_m} a_{sm}(x)\lim_{\varepsilon\to 0}{}^{L^p(\mathbb{R}^n)}\mathcal{R}_{sm,\varepsilon} f = \sum_{m=1}^{\infty}\sum_{s=1}^{g_m} a_{sm}(x)\mathcal{R}_{sm} f = \mathcal{K} f$$

and the same holds also for $\mathcal{C}_\varepsilon[\varphi, f]$. □

In the forthcoming applications to the regularity theory of elliptic PDEs, we need the following localized version of Theorem 4.8.

Theorem 4.9. *Let $\Omega \subset \mathbb{R}^n$ be a domain and $k(x; y)\colon\ \Omega \times \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ a* variable *CZ-kernel. For $f \in L^p(\Omega)$ with $p \in (1, \infty)$, $\varphi \in L^\infty(\Omega)$, $\varepsilon > 0$ and $x \in \Omega$ define*

$$\mathcal{K}_\varepsilon f(x) := \int_{\substack{|x-y|>\varepsilon\\ y\in\Omega}} k(x; x-y) f(y)\, dy,$$
$$\mathcal{C}_\varepsilon[\varphi, f](x) := \int_{\substack{|x-y|>\varepsilon\\ y\in\Omega}} k(x; x-y)\big[\varphi(x) - \varphi(y)\big] f(y)\, dy.$$

Then for each $f \in L^p(\Omega)$, $p \in (1, \infty)$, there exist $\mathcal{K}f, \mathcal{C}[\varphi, f] \in L^p(\Omega)$ such that

$$\lim_{\varepsilon\to 0}\big\|\mathcal{K}_\varepsilon f - \mathcal{K} f\big\|_{L^p(\Omega)} = \lim_{\varepsilon\to 0}\big\|\mathcal{C}_\varepsilon[\varphi, f] - \mathcal{C}[\varphi, f]\big\|_{L^p(\Omega)} = 0$$

and

$$\big\|\mathcal{K} f\big\|_{L^p(\Omega)} \le C\|f\|_{L^p(\Omega)},$$
$$\big\|\mathcal{C}[\varphi, f]\big\|_{L^p(\Omega)} \le C\|\varphi\|_*\|f\|_{L^p(\Omega)}$$

with a constant $C = C\left(n, p, \max_{|\beta|\le 2n}\big\|D_y^\beta k(x; y)\big\|_{L^\infty(\Omega\times\mathbb{S}^{n-1})}\right)$.

Proof. The function

$$\tilde{k}(x; y) := \begin{cases} k(x; y) & x \in \Omega,\ y \in \mathbb{R}^n \setminus \{0\},\\ 0 & x \notin \Omega,\ y \in \mathbb{R}^n \setminus \{0\}\end{cases}$$

satisfies the hypotheses of Theorem 4.8. Then, for $f \in L^p(\Omega)$ set

$$\tilde{f}(x) := \begin{cases} f(x) & x \in \Omega,\\ 0 & x \notin \Omega\end{cases}$$

and observe that

$$\mathcal{K}_\varepsilon f(x) = \int_{\substack{|x-y|>\varepsilon\\ y\in\Omega}} k(x; x-y) f(y)\, dy$$
$$= \int_{|x-y|>\varepsilon} \tilde{k}(x; x-y)\tilde{f}(y)\, dy =: \tilde{\mathcal{K}}_\varepsilon \tilde{f}(x) \quad \text{a.e. in } \Omega.$$

Theorem 4.8 ensures the existence of $\tilde{\mathcal{K}}\tilde{f} \in L^p(\mathbb{R}^n)$ such that

$$\lim_{\varepsilon\to 0}\big\|\tilde{\mathcal{K}}_\varepsilon \tilde{f} - \tilde{\mathcal{K}}\tilde{f}\big\|_{L^p(\mathbb{R}^n)} = 0.$$

Defining $\mathcal{K}f := \tilde{\mathcal{K}}\tilde{f}\big|_\Omega$, we have $\big\|\mathcal{K}_\varepsilon f - \mathcal{K}f\big\|_{L^p(\Omega)} \leq \big\|\tilde{\mathcal{K}}_\varepsilon \tilde{f} - \tilde{\mathcal{K}}\tilde{f}\big\|_{L^p(\mathbb{R}^n)}$, whence

$$\lim_{\varepsilon\to 0} \big\|\mathcal{K}_\varepsilon f - \mathcal{K}f\big\|_{L^p(\Omega)} = 0$$

and

$$\big\|\mathcal{K}f\big\|_{L^p(\Omega)} \leq \big\|\tilde{\mathcal{K}}\tilde{f}\big\|_{L^p(\mathbb{R}^n)} \leq C\big\|\tilde{f}\big\|_{L^p(\mathbb{R}^n)} = C\big\|f\big\|_{L^p(\Omega)}.$$

Similar arguments apply also to $\mathcal{C}[\varphi, f]$. □

Remark 4.10. Let $\varphi, \psi \in L^\infty(\mathbb{R}^n)$ and $\varphi = \psi$ a.e. in a domain Ω. Then $\mathcal{C}_\varepsilon[\varphi, f] = \mathcal{C}_\varepsilon[\psi, f]$ a.e. in Ω for each $f \in L^p(\Omega)$, whence $\mathcal{C}[\varphi, f] = \mathcal{C}[\psi, f]$ a.e. in Ω as well.

An outgrowth of the previous theorems is the following result which is central in the study of the regularity properties of elliptic PDEs with discontinuous coefficients.

Theorem 4.11. *Let k be as in Theorem 4.9 and assume $a \in VMO \cap L^\infty(\mathbb{R}^n)$ with VMO-modulus γ_a.*

Then for each $\varepsilon > 0$ there exists a $\rho_0 = \rho_0(\varepsilon, \gamma_a) > 0$ such that for each $\rho \in (0, \rho_0)$ and each ball $B_\rho \subset \Omega$ one has

$$\big\|\mathcal{C}[a, f]\big\|_{L^p(B_\rho)} \leq C\varepsilon\|f\|_{L^p(B_\rho)} \quad \forall f \in L^p(B_\rho)$$

with a constant C depending on n, p and the constant M from (4.10).

Proof. By the Sarason characterization VMO is the BMO-closure of BUC. Therefore, there exists a bounded and uniformly continuous function $\varphi\colon\ \mathbb{R}^n \to \mathbb{R}$ such that

$$\|a - \varphi\|_* < \frac{\varepsilon}{2}.$$

Let $\omega_\varphi(\rho)$ be the modulus of continuity of φ and choose $\rho_0 > 0$ in a way that

$$\omega_\varphi(\rho_0) < \frac{\varepsilon}{2}.$$

Fix a ball $B_\rho \subset \Omega$ with $\rho \in (0, \rho_0)$, centered at a point $x_0 \in \Omega$, and define

$$\psi(x) := \begin{cases} \varphi(x) & \text{if } x \in B_\rho, \\ \varphi\left(x_0 + \rho\dfrac{x - x_0}{|x - x_0|}\right) & \text{if } x \in \mathbb{R}^n \setminus B_\rho. \end{cases}$$

It is clear that $\psi \in \mathrm{BUC}(\mathbb{R}^n)$ and $\operatorname{osc}_{\mathbb{R}^n} \psi = \operatorname{osc}_{B_\rho} \varphi$. Further on,

$$\big\|\mathcal{C}[a, f]\big\|_{L^p(B_\rho)} \leq \big\|\mathcal{C}[a - \varphi, f]\big\|_{L^p(B_\rho)} + \big\|\mathcal{C}[\varphi, f]\big\|_{L^p(B_\rho)}.$$

It follows from Theorem 4.9 that

$$\big\|\mathcal{C}[a - \varphi, f]\big\|_{L^p(B_\rho)} \leq C\|a - \varphi\|_*\|f\|_{L^p(B_\rho)},$$

while Remark 4.10 gives

$$\big\|\mathcal{C}[\varphi, f]\big\|_{L^p(B_\rho)} = \big\|\mathcal{C}[\psi, f]\big\|_{L^p(B_\rho)} \leq C\|\psi\|_*\|f\|_{L^p(B_\rho)} \leq C\omega_\varphi(\rho)\|f\|_{L^p(B_\rho)}.$$

Therefore

$$\big\|\mathcal{C}[a, f]\big\|_{L^p(B_\rho)} \leq C\Big(\|a - \varphi\|_* + \omega_\varphi(\rho)\Big)\|f\|_{L^p(B_\rho)} \leq C\varepsilon\|f\|_{L^p(B_\rho)}$$

as desired. □

5. Second-order Elliptic PDEs with Discontinuous Coefficients

5.1. Interior $W^{2,p}$-estimates. Let $\Omega \subset \mathbb{R}^n$ be an open set and consider the linear, second-order elliptic equation in non-divergence form

$$\mathcal{L}u \equiv \sum_{i,j=1}^{n} a^{ij}(x) D_{x_i x_j} u = f(x) \quad \text{a.e. in } \Omega. \tag{5.1}$$

A *strong solution* to (5.1) is a twice weakly differentiable function $u\colon\ \Omega \to \mathbb{R}$ satisfying (5.1) *almost everywhere* in Ω.

Regarding the equation (5.1), we dispose of the *Schauder theory* that solves the regularity problem in the framework of the Hölder spaces. Namely, if $u \in C^2_{\mathrm{loc}}(\Omega)$ is a *classical solution* of (5.1) with Hölder continuous coefficients and right-hand side, $a^{ij}, f \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$, $\alpha \in (0,1)$, then $u \in C^{2,\alpha}_{\mathrm{loc}}(\Omega)$, and the same result holds true also *globally* (up to the smooth boundary $\partial\Omega$) in $\overline{\Omega}$ (see [7, Chapters 4 and 6]).

If a^{ij} are *merely continuous,* the Schauder theory is no more valid, and relevant regularity theory has been developed by *Calderón and Zygmund* in the settings of the L^p-spaces. The essence of that theory asserts that if u is a strong solution to (5.1) with $a^{ij} \in C^0_{\mathrm{loc}}(\Omega)$ and $f \in L^p_{\mathrm{loc}}(\Omega)$ with $p \in (1,\infty)$, then $u \in W^{2,p}_{\mathrm{loc}}(\Omega)$ (cf. [7, Chapter 9]). Moreover, a global $W^{2,p}(\Omega)$ result holds if $\partial\Omega$ is smooth enough.

Our aim here will be to develop *L^p-regularity theory* for (5.1) with *discontinuous* principal coefficients a^{ij}, following the works of Chiarenza–Frasca–Longo [3, 4]. Precisely, we will show that discontinuity of a^{ij}'s, controlled in terms of VMO, is a sufficient condition ensuring the *L^p-elliptic regularizing property* of the operator $\mathcal{L}$, that is, $\mathcal{L}u \in L^p(\Omega)$ yields $u \in W^{2,p}(\Omega)$.

In what follows, we will assume *uniform ellipticity* of the operator $\mathcal{L}$ and will allow *discontinuous* coefficients with discontinuity measured in VMO :

$$\begin{cases} a^{ij} \in VMO \cap L^\infty(\Omega), \\ a^{ij}(x) = a^{ji}(x) \quad \text{a.e. in } \Omega, \\ \exists \lambda > 0\colon \quad \lambda^{-1}|\xi|^2 \le \sum_{i,j=1}^n a^{ij}(x)\xi_i\xi_j \le \lambda|\xi|^2 \quad \text{a.e. } \Omega,\ \forall \xi \in \mathbb{R}^n. \end{cases} \tag{5.2}$$

Let $B \subset \Omega$ be any ball. Consider the function

$$\Gamma(x;y) := \begin{cases} \dfrac{1}{n(2-n)\omega_n\big(\det\{a^{ij}(x)\}\big)^{1/2}} \left(\displaystyle\sum_{i,j=1}^{n} A^{ij}(x) y_i y_j\right)^{\frac{2-n}{2}} & \text{if } n \ge 3, \\[2ex] \dfrac{1}{2\pi\big(\det\{a^{ij}(x)\}\big)^{1/2}} \log \left(\displaystyle\sum_{i,j=1}^{2} A^{ij}(x) y_i y_j\right)^{\frac{1}{2}} & \text{if } n = 2, \end{cases}$$

for a.a. $x \in B$, $\forall y \in \mathbb{R}^n \setminus \{0\}$ and where $\big\{A^{ij}(x)\big\}$ is the inverse matrix $\big\{a^{ij}(x)\big\}^{-1}$.
Note: If $a^{ij} = \delta_{ij}$ then $\Gamma(x;y) \equiv \Gamma_\Delta(y)$!

For each fixed $x_0 \in B$, $\Gamma(x_0; y)$ is the *fundamental solution* of the operator

$$\mathcal{L}_0 = \sum_{i,j=1}^{n} a^{ij}(x_0) D_{ij}$$

with coefficients "freezed" at x_0.

We set also

$$\Gamma_i(x; y) := \frac{\partial}{\partial y_i}\Gamma(x; y), \quad \Gamma_{ij}(x; y) := \frac{\partial^2}{\partial y_i \partial y_j}\Gamma(x; y)$$

and observe that $\Gamma_{ij}(x; y)$, $i, j = 1, \ldots, n$, are *variable CZ-kernels* with respect to $y \in \mathbb{R}^n \setminus \{0\}$. In fact, the $(-n)$-degree homogeneity is obvious, while their integral over $\mathbb{S}^{n-1}$ is zero because Γ_{ij} are first derivatives of Γ_i that are homogeneous functions of degree $1 - n$ (a direct proof follows by application of the Divergence Theorem to Γ_i in the corona $\{x \in \mathbb{R}^n : \ 1 \le |x| \le 2\}$).

We will obtain now a representation formula for the second derivatives of any strong solution to the equation $\mathcal{L}u = f$ in terms of *singular integrals* with density f and their *commutators* with densities the very same second derivatives $D_{x_h x_k} u$.

Theorem 5.1 (**Interior representation formula).** *Under the hypotheses* (5.2), *let* $u \in W^{2,p}_0(B)$ *with* $p \in (1, \infty)$.

Then, for a.a. $x \in B$ *we have*

$$D_{ij}u(x) = \text{P.V.} \int_B \Gamma_{ij}(x; x - y)\mathcal{L}u(y)\, dy + \text{P.V.} \int_B \Gamma_{ij}(x; x - y) \sum_{h,k=1}^{n} \big[a^{hk}(x) - a^{hk}(y)\big] D_{hk}u(y)\, dy + \mathcal{L}u(x) \int_{\mathbb{S}^{n-1}} \Gamma_i(x; z) z_j \, d\sigma_z. \tag{5.3}$$

Proof. We will get (5.3) assuming $u \in C^\infty_0(B)$. The general case then follows by density arguments using Theorem 4.9.

Fix a $x_0 \in B$. Since $\Gamma(x_0; x - y)$ is the fundamental solution of the *constant* coefficients operator $\mathcal{L}_0 = \sum_{i,j=1}^{n} a^{ij}(x_0) D_{ij}$, and

$$\mathcal{L}_0 u = \big(\mathcal{L}_0 - \mathcal{L}\big)u + \mathcal{L}u,$$

we have

$$u(x) = \int_B \Gamma(x_0; x - y) \underbrace{\left\{ \sum_{h,k=1}^{n} \big[a^{hk}(x_0) - a^{hk}(y)\big] D_{hk}u(y) + \mathcal{L}u(y) \right\}}_{=\mathcal{L}_0 u(y)} dy$$

for all $x \in B$ and for each fixed $x_0 \in B$.

Define now

$$g(y) := \begin{cases} \mathcal{L}_0 u(y) & \text{if } y \in B, \\ 0 & \text{if } y \notin B, \end{cases}$$

and note that $g \in C_0^\infty(\mathbb{R}^n)$. Thus

$$u(x) = \int_{\mathbb{R}^n} \Gamma(x_0; x-y) g(y)\, dy = \big(\Gamma * g\big)(x) \quad \forall x \in \mathbb{R}^n$$

and

$$\xi_i \xi_j \mathcal{F}(u) = \xi_i \xi_j \mathcal{F}(\Gamma)\mathcal{F}(g) \quad \forall \xi \in \mathbb{R}^n, \tag{5.4}$$

with the Fourier transform

$$\mathcal{F}(\varphi)(\xi) = \int_{\mathbb{R}^n} \varphi(x) e^{-2\pi i\, x\cdot\xi}\, dx.$$

Our aim is to show that there exist constants $c_{ij}(x_0)$ such that

$$-4\pi^2 \xi_i \xi_j \mathcal{F}(\Gamma) = \mathcal{F}\Big(\text{P.V. } \Gamma_{ij}(x_0; \cdot) + c_{ij}(x_0)\delta_0\Big) \quad \text{in } \mathcal{S}' \tag{5.5}$$

where δ_0 is the Dirac δ supported at the origin and $\mathcal{S}'$ is the Schwartz class of distributions.

For, for each $\varphi \in \mathcal{S}$ one has

$$\begin{aligned} \big\langle \xi_i\xi_j \mathcal{F}(\Gamma), \varphi\big\rangle = \big\langle \Gamma, \mathcal{F}(\xi_i\xi_j\varphi)\big\rangle &= -\frac{1}{4\pi^2}\left\langle \Gamma, \frac{\partial^2}{\partial y_i \partial y_j}\mathcal{F}(\varphi)\right\rangle \\ &= -\frac{1}{4\pi^2}\left\langle \frac{\partial^2 \Gamma}{\partial y_i \partial y_j}, \mathcal{F}(\varphi)\right\rangle. \end{aligned}$$

Further on,

$$\begin{aligned} \left\langle \frac{\partial \Gamma_i}{\partial y_j}, \varphi\right\rangle &= -\left\langle \Gamma_i, \frac{\partial}{\partial y_j}\varphi\right\rangle = -\int_{\mathbb{R}^n} \Gamma_i(x_0; y)\frac{\partial \varphi}{\partial y_j}(y)\, dy \\ &= -\lim_{\varepsilon\to 0}\int_{|y|>\varepsilon} \Gamma_i(x_0; y)\frac{\partial \varphi}{\partial y_j}(y)\, dy \\ &= \lim_{\varepsilon\to 0}\int_{|y|>\varepsilon} \frac{\partial \Gamma_i}{\partial y_j}(x_0; y)\varphi(y)\, dy + \lim_{\varepsilon\to 0}\int_{|y|=\varepsilon} \Gamma_i(x_0; y)\varphi(y)\frac{y_j}{|y|}\, d\sigma_y \\ &= \big\langle \text{P.V. } \Gamma_{ij}, \varphi\big\rangle + \lim_{\varepsilon\to 0}\int_{|z|=1} \Gamma_i(x_0; z)\varphi(\varepsilon z) z_j\, d\sigma_z \\ &= \big\langle \text{P.V. } \Gamma_{ij}, \varphi\big\rangle + \varphi(0)\int_{|z|=1} \Gamma_i(x_0; z) z_j\, d\sigma_z, \end{aligned}$$

and it suffices to set

$$c_{ij}(x_0) := \int_{|z|=1} \Gamma_i(x_0; z) z_j\, d\sigma_z$$

in order to get (5.5).

Recall now

$$\mathcal{F}\big(D_{ij}u\big)(\xi) = -4\pi^2 \xi_i\xi_j \mathcal{F}(u)(\xi)$$

and use (5.5) to obtain from (5.4) that

$$\begin{aligned} \mathcal{F}\big(D_{ij}u\big) &= \mathcal{F}\big(\text{P.V. } \Gamma_{ij}(x_0; \cdot) + c_{ij}(x_0)\delta_0\big)\cdot \mathcal{F}(g) \\ &= \mathcal{F}\Big(\big\{\text{P.V. } \Gamma_{ij}(x_0; \cdot) + c_{ij}(x_0)\delta_0\big\} * g\Big). \end{aligned}$$

So, we get by the Fourier inversion formula that

$$D_{ij}u(x) = \lim_{\varepsilon\to 0}\int_{|x-y|>\varepsilon}\Gamma_{ij}(x_0; x-y)g(y)\,dy + c_{ij}(x_0)g(x)$$

in $\mathcal{S}'$. SInce $g, u \in C^\infty$ and $\lim_{\varepsilon\to 0}\mathcal{K}_\varepsilon f = \mathcal{K}f$ in $C^{0,\alpha}$ (see Theorem 4.2 and the note after it), the same formula holds true pointwisely in B.

Letting finally $x = x_0$ and applying density arguments we get (5.3). □

We are in a position now to study the regularity problem for (5.1). Recall that $a^{ij} \in \mathrm{VMO}$ and set

$$\gamma_a(\rho) := \left(\sum_{i,j=1}^{n}\gamma^2_{a^{ij}}(\rho)\right)^{1/2}$$

and

$$M := \max_{i,j}\max_{|\beta|=2n}\left\|\frac{\partial^\beta\Gamma_{ij}(x;y)}{\partial y^\beta}\right\|_{L^\infty(\Omega\times\mathbb{S}^{n-1})} \qquad \text{(cf. (4.10))}.$$

Theorem 5.2 (**Local L^p-estimate for the second derivatives).** *Under the hypotheses* (5.2)*, for each* $p \in (1,\infty)$ *there exist constants* C *and* ρ_0*, depending on* n, p, λ, M *and* γ_a*, such that*

$$\|D^2u\|_{L^p(B_\rho)} \le C\|\mathcal{L}u\|_{L^p(B_\rho)}$$

$\forall B_\rho \Subset \Omega$, $\forall \rho < \rho_0$, *and* $\forall u \in W^{2,p}_0(B_\rho)$.

Proof. Taking L^p-norms on the both sides of (5.3) gives

$$\begin{aligned}\|D^2u\|_{L^p(B_\rho)} \le{}& \underbrace{\left\|\mathrm{P.V.}\int_{B_\rho}\Gamma_{ij}(x;x-y)\mathcal{L}u(y)\,dy\right\|_{L^p(B_\rho)}}_{N_1}\\ &+\underbrace{\left\|\mathrm{P.V.}\int_{B}\Gamma_{ij}(x;x-y)\sum_{h,k=1}^{n}\left[a^{hk}(x)-a^{hk}(y)\right]D_{hk}u(y)\,dy\right\|_{L^p(B_\rho)}}_{N_2}\\ &+\underbrace{\left\|\mathcal{L}u(x)\int_{\mathbb{S}^{n-1}}\Gamma_i(x;z)z_j\,d\sigma_z\right\|_{L^p(B_\rho)}}_{N_3} \qquad \forall B_\rho \Subset \Omega.\end{aligned}$$

It follows from Theorem 4.9 that

$$N_1 \le C(n,p,M)\|\mathcal{L}u\|_{L^p(B_\rho)},$$

while Theorem 4.11 and $a^{ij} \in \mathrm{VMO}$ ensure that $\forall\varepsilon > 0$ there exists $\rho_0 > 0$ such that

$$N_2 \le C\varepsilon\|D^2u\|_{L^p(B_\rho)} \quad \forall \rho < \rho_0.$$

Finally, the surface integral in (5.3) is bounded whence

$$N_3 \le C\big\|\mathcal{L}u\big\|_{L^p(B_\rho)}.$$

This way, choosing $\varepsilon > 0$ small enough to move $C\varepsilon\big\|D^2u\big\|_{L^p(B_\rho)}$ on the left-hand side, we get the claim. □

Theorem 5.3 (**Elliptic regularizing property, Interior $W^{2,p}$-estimate).** *Assume* (5.2).

Then, for all $1 < q \le p < \infty$ *and all* $u \in W^{2,q}_{\mathrm{loc}}(\Omega)$ *such that* $\mathcal{L}u \in L^p_{\mathrm{loc}}(\Omega)$ *it follows*

$$u \in W^{2,p}_{\mathrm{loc}}(\Omega).$$

Moreover, for all open $\Omega' \Subset \Omega'' \Subset \Omega$ *there exists a constant* C*, depending on* $n, p, \lambda, \gamma_a, M$ *and* $\operatorname{dist}(\Omega', \partial\Omega'')$ *such that*

$$\|u\|_{W^{2,p}(\Omega')} \le C\Big(\|u\|_{L^p(\Omega'')} + \big\|\mathcal{L}u\big\|_{L^p(\Omega'')}\Big).$$

Proof. For $i, j, k, h = 1, \dots, n$ define

$$S_{ijhk}(f)(x) := \text{P.V.}\int_{B_\rho} \Gamma_{ij}(x; x-y)\big[a^{hk}(x) - a^{hk}(y)\big] f(y)\, dy$$

where $B_\rho \Subset \Omega$ and $f \in L^r(B_\rho)$ with $r \in (1, \infty)$.

Fix ρ_0 as in Theorem 4.11 to be so small to have

$$\sum_{i,j,k,h} \big\|S_{ijhk}\big\|_{L^r(B_\rho)\to L^r(B_\rho)} < 1$$

with the operator norms taken in the space of all linear operators from $L^r(B_\rho)$ into itself with $r \in [q, p]$.

Fix a ball $B \Subset \Omega$ of radius less than ρ_0, take a cut-off function $\beta \in C_0^\infty(B)$, $\beta \equiv 1$ on $B' \Subset B$, and set

$$v := \beta u \in W_0^{2,q}(B).$$

Straightforward calculations give

$$\mathcal{L}v = \underbrace{u}_{\in W^{2,q}}(\mathcal{L}\beta) + 2\sum_{i,j=1}^n a^{ij}(x) D_i\beta\, \underbrace{D_j u}_{\in W^{1,q}} + \beta(\underbrace{\mathcal{L}u}_{\in L^p}).$$

If follows from the Sobolev embedding theorem that there is a $q_1 \in (q, p]$ such that

$$\mathcal{L}v \in L^{q_1}(B),$$

while (5.3) yields

$$D_{ij}v(x) = \text{P.V.}\int_B \Gamma_{ij}(x; x-y) \sum_{h,k=1}^n \big[a^{hk}(x) - a^{hk}(y)\big] D_{hk}v(y)\, dy \tag{5.6}$$

$$+ \text{P.V.}\int_B \Gamma_{ij}(x; x-y)\mathcal{L}v(y)\, dy + c_{ij}(x)\mathcal{L}v(x)$$

with

$$\left\|c_{ij}\right\|_{L^\infty(\Omega)} \le C(n,\lambda).$$

Define now the matrix $\mathbf{H} = \left\{H_{ij}\right\}_{i,j=1}^n$ with entries

$$H_{ij}(x) = \text{P.V.} \int_B \Gamma_{ij}(x; x-y)\mathcal{L}v(y)\, dy + c_{ij}(x)\mathcal{L}v(x)$$

and note that $H_{ij} \in L^{q_1}(B)$ by means of Theorem 4.9 and since $\mathcal{L}v \in L^{q_1}$.

Let $r \in [q, q_1]$ and $\mathbf{w} = \{w_{ij}\}_{i,j=1}^n \in \left[L^r(B)\right]^{n^2}$. Define the mapping

$$\mathcal{T}\colon\ \left[L^r(B)\right]^{n^2} \longrightarrow \left[L^r(B)\right]^{n^2}$$

by

$$\mathcal{T}\mathbf{w} = \left\{(\mathcal{T}\mathbf{w})_{ij}\right\}_{i,j=1}^n = \left\{\sum_{h,k=1}^n S_{ijhk}(w_{hk})(x) + H_{ij}(x)\right\}_{i,j=1}^n .$$

We have

$$\begin{aligned}\left\|\mathcal{T}\mathbf{w}' - \mathcal{T}\mathbf{w}''\right\|_{[L^r(B)]^{n^2}} &= \left\|\sum_{h,k=1}^n S_{ijhk}\left(w'_{hk} - w''_{hk}\right)\right\|_{[L^r(B)]^{n^2}} \\ &\le \underbrace{\sum_{i,j,k,h} \left\|S_{ijhk}\right\|_{L^r(B)\to L^r(B)}}_{<1} \cdot \left\|\mathbf{w}' - \mathbf{w}''\right\|_{[L^r(B)]^{n^2}} ,\end{aligned}$$

whence $\mathcal{T}$ is a *contraction mapping* from $\left[L^r(B)\right]^{n^2}$ into itself for *each* $r \in [q, q_1]$. Therefore, $\mathcal{T}$ has a *unique fixed point in all* $\left[L^r(B)\right]^{n^2}$ with $r \in [q, q_1]$.

Let $\overline{\mathbf{w}} = \left\{\overline{w}_{ij}\right\}_{i,j=1}^n$ be the fixed point of $\mathcal{T}$ in $\left[L^{q_1}(B)\right]^{n^2} \subset \left[L^q(B)\right]^{n^2}$. It follows from (5.6) that also $\left\{D^2v\right\}$ is a fixed point of $\mathcal{T}$ in $\left[L^q(B)\right]^{n^2}$ and the uniqueness yields

$$\left\{D^2 v\right\} = \overline{\mathbf{w}} = \left\{\overline{w}_{ij}\right\}_{i,j=1}^n \in L^{q_1}(B).$$

If $q_1 = p$ we have $v \in W^{2,p}(B)$, and since $B' \Subset B$ is arbitrary, we get the claim

$$u \in W^{2,p}_{\text{loc}}(\Omega).$$

If $q_1 < p$, then the above arguments apply by iteration.

Finally, the interior $W^{2,p}$-estimate follows by covering and using Theorem 5.2. □

5.2. Boundary $W^{2,p}$-estimates. We will consider first of all equations in the half-space

$$\mathbb{R}^n_+ := \left\{x = (x', x_n)\colon\ x' \in \mathbb{R}^{n-1},\ x_n > 0\right\}.$$

For each $x \in \mathbb{R}^n_+$ we let $\tilde{x} := (x', -x_n)$.

5.2.1. L^p-boundedness of boundary integral operators and commutators. We start with the following auxiliary result, asserting L^p-boundedness of *non-singular* integrals.

Lemma 5.4. *Let $f \in L^p(\mathbb{R}^n_+)$ with $p \in (1,\infty)$. For $x \in \mathbb{R}^n_+$ set*

$$\widetilde{\mathcal{K}}f(x) := \int_{\mathbb{R}^n_+} \frac{f(y)}{|\tilde{x}-y|^n}dy.$$

Then

$$\big\|\widetilde{\mathcal{K}}f\big\|_{L^p(\mathbb{R}^n_+)} \leq C(n,p)\|f\|_{L^p(\mathbb{R}^n_+)}.$$

Proof. Define

$$\begin{aligned} I(x_n) &:= \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}^n_+} \frac{|f(y)|}{\big(|x'-y'|^2+(x_n+y_n)^2\big)^{n/2}}dy \right)^p dx' \\ &= \int_{\mathbb{R}^{n-1}} \left(\int_0^\infty \left(\int_{\mathbb{R}^{n-1}} \frac{|f(y)|}{\big(|x'-y'|^2+(x_n+y_n)^2\big)^{n/2}}dy' \right) dy_n \right)^p dx'. \end{aligned}$$

Setting

$$\varphi(y_n) := \left(\int_{\mathbb{R}^{n-1}} |f(y',y_n)|^p dy' \right)^{1/p},$$

we have by the Minkowski[1] and Young[2] inequalities

$$\begin{aligned} I(x_n) &\leq \left[\int_0^\infty \left(\int_{\mathbb{R}^{n-1}} |f(y',y_n)|^p dy' \right)^{1/p} \left(\int_{\mathbb{R}^{n-1}} \frac{dy'}{\big(|y'|^2+(x_n+y_n)^2\big)^{n/2}} \right) dy_n \right]^p \\ &= \left(\int_0^\infty \frac{\varphi(y_n)}{x_n+y_n} \right)^p \left(\int_{\mathbb{R}^{n-1}} \frac{dt}{\big(|t|^2+1\big)^{n/2}} \right)^p. \end{aligned}$$

Integration in $x_n \in (0,\infty)$ and another application of the Minkowski inequality gives finally

$$\begin{aligned} \big\|\widetilde{\mathcal{K}}f\big\|^p_{L^p(\mathbb{R}^n_+)} &\leq C(n,p) \int_0^\infty \left(\int_0^\infty \frac{\varphi(sx_n)}{1+s} ds \right)^p dx_n \\ &\leq C(n,p) \left(\int_0^\infty \left(\int_0^\infty \left(\frac{\varphi(sx_n)}{1+s} \right)^p dx_n \right)^{1/p} ds \right)^p \\ &= C(n,p) \left(\int_0^\infty \frac{ds}{(1+s)s^{1/p}} \right)^p \|f\|_{L^p(\mathbb{R}^n_+)}. \end{aligned}$$

□

Following the proof of Theorem 4.4, we get the following result regarding *non-singular commutators.*

---

[1] $\left(\int_{S_1}\left|\int_{S_2} F(x,y)\,\mu_1(dx)\right|^p \mu_2(dy)\right)^{1/p} \leq \int_{S_1}\left(\int_{S_2} |F(x,y)|^p\,\mu_2(dy)\right)^{1/p} \mu_1(dx)$ with measure spaces (S_1,μ_1) and (S_2,μ_2) and measurable function $F\colon\ S_1\times S_2 \to \mathbb{R}$

[2] $\|f*g\|_{L^r} \leq \|f\|_{L^p}\|g\|_{L^q}$ with $p,q,r \in [1,\infty]$ and such that $1/p+1/q = 1+1/r$

Lemma 5.5. *Let $f \in L^p(\mathbb{R}^n_+)$ with $p \in (1,\infty)$ and $\varphi \in L^\infty(\mathbb{R}^n_+)$. For $x \in \mathbb{R}^n_+$ define*

$$\widetilde{\mathcal{C}}[\varphi, f](x) := \int_{\mathbb{R}^n_+} \frac{|\varphi(x) - \varphi(y)|}{|\tilde{x} - y|^n} f(y)\, dy.$$

Then

$$\left\|\widetilde{\mathcal{C}}[\varphi, f]\right\|_{L^p(\mathbb{R}^n_+)} \le C(n,p)\|\varphi\|_* \|f\|_{L^p(\mathbb{R}^n_+)}.$$

Setting

$$B^+_\rho := \left\{(x', x_n) \in \mathbb{R}^n\colon\ |x| < \rho,\ x_n > 0\right\}$$

for the half-ball centered at $\mathbb{R}^{n-1}$ and lying in $\mathbb{R}^n_+$, we obtain as in Theorem 4.11

Lemma 5.6. *Under the hypotheses of Lemma* 5.5, *let $\varphi \in \mathrm{VMO} \cap L^\infty(\mathbb{R}^m_+)$. Then for each $\varepsilon > 0$ there exists $\rho_0 = \rho_0(\varepsilon, \gamma_\varphi)$ such that for each $\rho \in (0, \rho_0)$ it holds*

$$\left\|\widetilde{\mathcal{C}}[\varphi, f]\right\|_{L^p(B^+_\rho)} \le C\varepsilon \|f\|_{L^p(B^+_\rho)}.$$

5.2.2. Local boundary $W^{2,p}$-estimates. Let $\overline{W}^{2,p}(B^+_\rho)$ be the closure in $W^{2,p}$ of the space

$$\overline{\mathcal{C}} := \Big\{u\colon\ B^+_\rho \to \mathbb{R}\colon\quad u \text{ is the restriction to } B^+_\rho \text{ of a function in } C^\infty_0(B_\rho) \text{ such that } u(x', 0) = 0\Big\}.$$

We set

$$\mathbf{b}(x) = \left(a^{1n}(x), \dots, a^{nn}(x)\right)$$

for the last row/column of the coefficient matrix $\left\{a^{ij}(x)\right\}^n_{i,j=1}$ and define the transformation $\mathbf{T}\colon\ \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to \mathbb{R}^n_-$ by

$$\mathbf{T}(x; y) := x - \frac{2x_n}{a^{nn}(y)} \mathbf{b}(y).$$

If $\mathcal{L} \equiv \Delta$ we have $a^{ij} = \delta_{ij}$ and $\mathbf{T}(x; y) = \tilde{x}$ which means that $\mathbf{T}$ is a sort of "generalized reflection" subordinated to the coefficients matrix $\left\{a^{ij}(x)\right\}^n_{i,j=1}$.

Setting $e_n = \big(0, 0, \dots, 0, 1\big)$ for the n-th coordinate vector we define also

$$\mathbf{T}(x) := \mathbf{T}(x; x), \quad \mathbf{B}(y) := \mathbf{T}(e_n; y)$$

and note that $\mathbf{B}(y) = \big(0, 0, \dots, 0, -1\big)$ when $\mathcal{L} \equiv \Delta$.

The next result shows that the distance between $\mathbf{T}(x)$ and y is comparable with that between $\tilde{x}$ and y.

Lemma 5.7. *There exists a constant $C = C(n, \lambda)$ such that*

$$C^{-1}|\tilde{x} - y| \le \big|\mathbf{T}(x) - y\big| \le C|\tilde{x} - y| \quad \forall y \in \mathbb{R}^n_+, \text{ for a.a. } x \in \mathbb{R}^n_+.$$

Proof. It is clear that $\mathbf{T}(x) - y = \left(x' - y' - \dfrac{2x_n}{a^{nn}(x)} a'^n(x), -x_n - y_n\right)$, whence $\big|\mathbf{T}(x) - y\big| \ge x_n + y_n \ge x_n$. On the other hand,

$$\big|\mathbf{T}(x) - \tilde{x}\big| = 2x_n \frac{|\mathbf{b}(x)|}{a^{nn}(x)} \le c(n, \lambda) x_n$$

and this gives

$$|\tilde{x}-y| \le \big|\mathbf{T}(x)-\tilde{x}\big| + \big|\mathbf{T}(x)-y\big| \le \big(1+c(n,\lambda)\big)\big|\mathbf{T}(x)-y\big|.$$

Similarly,

$$\big|\mathbf{T}(x)-y\big| \le \big|\mathbf{T}(x)-\tilde{x}\big| + |\tilde{x}-y| \le c(n,\lambda)x_n + |\tilde{x}-y| \le \big(1+c(n,\lambda)\big)|\tilde{x}-y|$$

since $|\tilde{x}-y| \ge x_n$ as well. □

The boundary $W^{2,p}$-estimates will be obtained as in the interior case, making use of a representation formula for the second derivatives of the strong solution to $\mathcal{L}u = f$.

Theorem 5.8 (Boundary representation formula). *Assume* (5.2) *and let* $u \in \overline{W}^{2,p}(B_\rho^+)$ *with* $p \in (1,\infty)$.

Then, for a.a. $x \in B_\rho^+$ *we have*

$$
\begin{aligned}
(5.7)\qquad D_{ij}u(x) = {} & \text{P.V.}\int_{B_\rho^+} \Gamma_{ij}(x;x-y)\mathcal{L}u(y)\,dy \\
& + \text{P.V.}\int_{B_\rho^+} \Gamma_{ij}(x;x-y)\sum_{h,k=1}^{n}\big[a^{hk}(x)-a^{hk}(y)\big]D_{hk}u(y)\,dy \\
& + \mathcal{L}u(x)\int_{\mathbb{S}^{n-1}} \Gamma_i(x;z)z_j\,d\sigma_z \\
& + I_{ij}(x),
\end{aligned}
$$

where

$$
\begin{aligned}
I_{ij}(x) = {} & \int_{B_\rho^+} \Gamma_{ij}(x;\mathbf{T}(x)-y)\left\{\mathcal{L}u(y) + \sum_{h,k=1}^{n}\big[a^{hk}(x)-a^{hk}(y)\big]D_{hk}u(y)\right\}dy \\
& \textit{for } i,j = 1,\dots,n-1; \\
I_{in}(x) = {} & I_{ni}(x) \\
= {} & \int_{B_\rho^+}\left(\sum_{j=1}^{n}\Gamma_{ij}(x;\mathbf{T}(x)-y)B_j(x)\right) \\
& \qquad \times\left\{\mathcal{L}u(y) + \sum_{h,k=1}^{n}\big[a^{hk}(x)-a^{hk}(y)\big]D_{hk}u(y)\right\}dy \\
& \textit{for } i = 1,\dots,n-1; \\
I_{nn}(x) = {} & \int_{B_\rho^+}\left(\sum_{i,j=1}^{n}\Gamma_{ij}(x;\mathbf{T}(x)-y)B_i(x)B_j(x)\right) \\
& \qquad \times\left\{\mathcal{L}u(y) + \sum_{h,k=1}^{n}\big[a^{hk}(x)-a^{hk}(y)\big]D_{hk}u(y)\right\}dy,
\end{aligned}
$$

and where $\mathbf{B}(x) = \big(B_1(x),\dots,B_n(x)\big)$.

Proof. Let $x_0 \in B_\rho^+$ and $u \in \overline{\mathcal{C}}$. We use the half-space Green function

$$\Gamma(x_0; x-y) - \Gamma\big(x_0; \mathbf{T}(x;x_0) - y\big),$$

corresponding to the operator $\mathcal{L}_0 = \sum_{i,j=1}^n a^{ij}(x_0) D_{ij}$, to get

$$\begin{aligned} u(x) &= \int_{B_\rho^+} \Big(\Gamma(x_0; x-y) - \Gamma\big(x_0; \mathbf{T}(x;x_0) - y\big)\Big)\mathcal{L}_0 u(y)\, dy \\ &= \underbrace{\int_{B_\rho^+} \Gamma(x_0; x-y)\mathcal{L}_0 u(y)\, dy}_{I'} - \underbrace{\int_{B_\rho^+} \Gamma\big(x_0; \mathbf{T}(x;x_0) - y\big)\mathcal{L}_0 u(y)\, dy}_{I''}. \end{aligned}$$

Differentiating I' twice as in the proof of Theorem 5.1, and writing

$$\mathcal{L}_0 u = (\mathcal{L}_0 - \mathcal{L})u + \mathcal{L}u,$$

we obtain

$$\begin{aligned} D_{ij} I'(x) = \text{P.V.} &\int_{B_\rho^+} \Gamma_{ij}(x_0; x-y)\mathcal{L}u(y)\, dy \\ &+ \text{P.V.} \int_{B_\rho^+} \Gamma_{ij}(x_0; x-y) \sum_{h,k=1}^n \big[a^{hk}(x_0) - a^{hk}(y)\big] D_{hk} u(y)\, dy \\ &+ \mathcal{L}u(x) \int_{\mathbb{S}^{n-1}} \Gamma_i(x_0; z) z_j\, d\sigma_z \quad \forall i,j = 1,\dots,n,\ \forall x \in B_\rho^+. \end{aligned}$$

For what concerns I'', it is possible to differentiate under the sign of the integral because $\mathbf{T}(x;x_0) \in \mathbb{R}^n_-$ and $\big|\mathbf{T}(x;x_0) - y\big|$ is bounded away from 0.

This way, for $u \in \overline{\mathcal{C}}$, (5.7) follows by the setting $x_0 = x$, while density arguments give it for $u \in \overline{W}^{2,p}(B_\rho^+)$. □

Recall that

$$\gamma_a(\rho) = \left(\sum_{i,j=1}^n \gamma_{a^{ij}}^2(\rho)\right)^{1/2}, \quad M = \max_{i,j} \max_{|\beta|=2n} \left\| \frac{\partial^\beta \Gamma_{ij}(x;y)}{\partial y^\beta} \right\|_{L^\infty(\Omega\times\mathbb{S}^{n-1})}$$

as defined above.

Theorem 5.9 (**Boundary $W^{2,p}$-estimate).** *Assume* (5.2) *and let* $q, p \in (1,\infty)$ *with* $q \le p$.

Then there exists a constant $\rho_0 = \rho_0(n,p,q,\lambda,M,\gamma_a) > 0$ *such that for each* $\rho \in (0,\rho_0)$ *and each* $u \in \overline{W}^{2,q}(B_\rho^+)$ *with* $\mathcal{L}u \in L^p(B_\rho^+)$ *we have* $u \in W^{2,p}(B_\rho^+)$. *Moreover,*

$$\big\|D^2 u\big\|_{L^p(B_\rho^+)} \le C \big\|\mathcal{L}u\big\|_{L^p(B_\rho^+)}$$

with a constant $C = C(n,p,\lambda,M,\gamma_a)$.

Proof. We will proceed in the same manner as in the proofs of Theorems 5.2 and 5.3, making use of (5.7).

Thus, for $i, j, k, h = 1, \dots, n$ we set as before

$$S_{ijhk}(f)(x) := \text{P.V.} \int_{B_\rho^+} \Gamma_{ij}(x; x-y)\big[a^{hk}(x) - a^{hk}(y)\big] f(y)\, dy$$

Further on, define the *non-singular* commutators

$$\widetilde{S}_{ijhk}(f)(x) := \int_{B_\rho^+} \Gamma_{ij}(x; \mathbf{T}(x) - y)\big[a^{hk}(x) - a^{hk}(y)\big] f(y)\, dy$$

for $i, j = 1, \dots, n-1$ and $h, k = 1, \dots, n$;

$$\widetilde{S}_{inhk}(f)(x) := \int_{B_\rho^+} \left(\sum_{j=1}^{n} \Gamma_{ij}(x; \mathbf{T}(x) - y) B_j(x)\right) \big[a^{hk}(x) - a^{hk}(y)\big] f(y)\, dy$$

for $i = 1, \dots, n-1$ and $h, k = 1, \dots, n$;

$$\widetilde{S}_{nnhk}(f)(x) := \int_{B_\rho^+} \left(\sum_{i,j=1}^{n} \Gamma_{ij}(x; \mathbf{T}(x) - y) B_i(x) B_j(x)\right) \big[a^{hk}(x) - a^{hk}(y)\big] f(y)\, dy$$

for $h, k = 1, \dots, n$.

In view of Theorem 4.11 and Lemmas 5.6 and 5.7, we can fix ρ_0 so small to have

$$\sum_{i,j,k,h} \big\| S_{ijhk} + \widetilde{S}_{ijhk} \big\|_{L^r(B_\rho^+) \to L^r(B_\rho^+)} < 1$$

with the operator norms taken in the space of all linear operators from $L^r(B_\rho^+)$ into itself with $\rho \in (0, \rho_0)$ and $r \in [q, p]$.

Let $u \in \overline{W}^{2,q}(B_\rho^+)$ with $\mathcal{L}u \in L^p(B_\rho^+)$, $\rho \in (0, \rho_0)$, and set

$$h_{ij}(x) = \text{P.V.} \int_{B_\rho^+} \Gamma_{ij}(x; x-y) \mathcal{L}u(y)\, dy + \mathcal{L}u(x) \int_{|z|=1} \Gamma_i(x; z) z_j\, d\sigma_z + \widetilde{I}_{ij}(x)$$

with

$$\widetilde{I}_{ij}(x) := \begin{cases} \displaystyle\int_{B_\rho^+} \Gamma_{ij}(x; \mathbf{T}(x) - y) \mathcal{L}u(y)\, dy \\ \hfill \text{as } i, j = 1, \dots, n-1; \\ \displaystyle\int_{B_\rho^+} \left(\sum_{s=1}^{n} \Gamma_{is}(x; \mathbf{T}(x) - y) B_s(x)\right) \mathcal{L}u(y)\, dy \\ \hfill \text{as } i = 1, \dots, n-1,\ j = n; \\ \displaystyle\int_{B_\rho^+} \left(\sum_{s,t=1}^{n} \Gamma_{st}(x; \mathbf{T}(x) - y) B_s(x) B_t(x)\right) \mathcal{L}u(y)\, dy \\ \hfill \text{as } i = j = n. \end{cases}$$

Indeed, $h_{ij} \in L^p(B_\rho^+)$ by Theorem 4.9, Lemma 5.4 and Lemma 5.7.

Take $\mathbf{w} = \{w_{ij}\}_{i,j=1}^n \in \big[L^p(B_\rho^+)\big]^{n^2}$ and define the map

$$\mathcal{T}\colon\ \big[L^p(B_\rho^+)\big]^{n^2} \longrightarrow \big[L^p(B_\rho^+)\big]^{n^2}$$

by

$$\mathcal{T}\mathbf{w} = \left\{(\mathcal{T}\mathbf{w})_{ij}\right\}_{i,j=1}^{n} = \left\{ \sum_{h,k=1}^{n} \left(S_{ijhk} + \widetilde{S}_{ijhk} \right)(w_{hk})(x) + h_{ij}(x) \right\}_{i,j=1}^{n}.$$

We get as before that $\mathcal{T}$ is a *contraction* from $\left[L^r(B_\rho^+)\right]^{n^2}$ into itself for *each* $r \in [q,p]$, with a fixed point $\left\{D^2u\right\} = \left\{D_{ij}u\right\}_{i,j=1}^{n} \in \left[L^q(B_\rho^+)\right]^{n^2}$ as shows (5.7), that belongs also to $\left[L^p(B_\rho^+)\right]^{n^2}$ by the uniqueness.

The $L^p(B_\rho^+)$-estimate for the second derivatives follows as in Theorem 5.2 on the base of (5.7) and Lemmas 5.4, 5.5, 5.6 and 5.7. □

5.3. The Dirichlet problem. We will apply now the above results to *strong solvability* and *regularity* of solutions to the Dirichlet problem

$$\begin{cases} \mathcal{L}u \equiv \displaystyle\sum_{i,j=1}^{n} a^{ij}(x)D_{ij}u = f(x) \quad \text{a.e. } \Omega, \\ u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega), \end{cases} \tag{5.8}$$

where the zero boundary data are assumed to be attained in the sense of $W_0^{1,p}(\Omega)$.

Apart from the basic hypotheses (5.2) of uniform ellipticity and VMO-regularity of the principal coefficients a^{ij}, we will assume that the domain Ω is of *class* $C^{1,1}$.

Definition 5.10. A bounded domain $\Omega \subset \mathbb{R}^n$ and its boundary $\partial\Omega$ are said to be of class $C^{1,1}$ if at each point $x_0 \in \partial\Omega$ there is a ball $B = B(x_0)$ and a diffeomorphism $\mathbf{\Psi}\colon\ B \to D \subset \mathbb{R}^n$ with the properties:

- $\mathbf{\Psi}(B \cap \Omega) \subset \mathbb{R}_+^n$;
- $\mathbf{\Psi}(B \cap \partial\Omega) \subset \partial\mathbb{R}_+^n$;
- $\mathbf{\Psi} \in C^{1,1}(B)$, $\mathbf{\Psi}^{-1} \in C^{1,1}(D)$.

It should be noted that, taken any point $x_0 \in \partial\Omega$, there exist a neighbourhood $U(x_0)$ and a flattering diffeomorphism $\mathbf{\Psi}\colon\ U(x_0) \cap \Omega \to B_\rho^+$, so that $y = \mathbf{\Psi}(x)$ transforms the coefficients $a^{ij}(x)$ into

$$\tilde{a}^{ij}(y) = \sum_{h,k=1}^{n} a^{hk}\big(\mathbf{\Psi}^{-1}(y)\big)\frac{\partial}{\partial x_h}\Psi_i\big(\mathbf{\Psi}^{-1}(y)\big)\frac{\partial}{\partial x_k}\Psi_j\big(\mathbf{\Psi}^{-1}(y)\big), \quad y \in B_\rho^+.$$

Thus, $\tilde{a}^{ij}(y) = a^{ij}\big(\mathbf{\Psi}^{-1}(y)\big) \in \mathrm{VMO}$ with VMO-moduli $\gamma_{a^{ij}(\mathbf{\Psi}^{-1}(y))}$ satisfying

$$\frac{1}{C}\gamma_{a^{ij}} \le \gamma_{a^{ij}(\mathbf{\Psi}^{-1}(y))} \le C\gamma_{a^{ij}}$$

with $C = C\left(n, \left\|\mathbf{\Psi}\right\|_{C^{1,1}}\right)$.

The following result is a direct consequence of Theorems 5.3 and 5.9, after standard covering of $\overline{\Omega}$ and local flattering of $\partial\Omega$.

Theorem 5.11 (**Global $W^{2,p}$-estimate**)**.** *Assume* (5.2), $1 < q \leq p < \infty$ *and* $\partial\Omega \in C^{1,1}$. *Let* $u \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$ *be such that*

$$\mathcal{L}u \in L^p(\Omega).$$

Then

$$u \in W^{2,p}(\Omega)$$

and there exists a constant C, *depending on* $n, p, \lambda, \gamma_a, M, \partial\Omega$, *such that*

$$\|u\|_{W^{2,p}(\Omega)} \leq C\Big(\|u\|_{L^p(\Omega)} + \big\|\mathcal{L}u\big\|_{L^p(\Omega)}\Big). \tag{5.9}$$

An immediate consequence of Theorem 5.11, the Sobolev embedding theorem and the Morrey Lemma 2.3 is the following

Corollary 5.12. *The strong solution of* (5.8) *is Hölder continuous with exponent* $2 - n/p$ *if* $p > n/2$, *while its gradient is Hölder continuous with exponent* $1 - n/p$ *if* $p > n$.

The global L^p-regularizing property claimed in Theorem 5.11 allows to establish uniqueness of the strong solution to (5.8).

Theorem 5.13 (**Uniqueness for** (5.8))**.** *Under the hypotheses* (5.2), $p \in (1, \infty)$ *and* $\partial\Omega \in C^{1,1}$, *the solution of the homogeneous Dirichlet problem*

$$\Big\{\mathcal{L}u = 0 \quad a.e.\ \Omega, \qquad u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$$

is identically zero in $\overline{\Omega}$.

Proof. Since $0 \equiv f \in L^n(\Omega)$, Theorem 5.11 and the Sobolev embedding theorem imply

$$u \in W^{2,n}(\Omega) \cap C^0(\overline{\Omega})$$

whence $u \equiv 0$ by the Aleksandrov maximum principle[3]. □

It turns out that, thanks to the unicity, the term $\|u\|_{L^p(\Omega)}$ can be dropped from the righ-hand side of (5.9), straightening this way the global $W^{2,p}$-estimate.

Theorem 5.14. *Assume* (5.2), $\partial\Omega \in C^{1,1}$, $p \in (1, \infty)$ *and let* $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ *be a strong solution to* (5.8).

Then

$$\|u\|_{W^{2,p}(\Omega)} \leq C\big\|\mathcal{L}u\big\|_{L^p(\Omega)} \tag{5.10}$$

with $C = C\big(n, p, \lambda, \gamma_a, M, \partial\Omega\big)$.

[3] Let $\mathcal{L}u = f$ a.e. in a bounded domain Ω with $f \in L^n(\Omega)$ and $u \in C^0(\overline{\Omega}) \cap W^{2,n}_{\text{loc}}(\Omega)$. Then

$$\sup_{\Omega} |u| \leq \sup_{\partial\Omega} |u| + C\|f\|_{L^n(\Omega)}$$

with a constant C depending on n, λ and $\operatorname{diam}\Omega$ ([7, Theorem 9.1])

Proof. We argue by contradiction in order to prove (5.10). So, if (5.10) is false then there exist a sequence of uniformly elliptic operators

$$\mathcal{L}^{(m)} \equiv \sum_{i,j=1}^{n} a^{ij}_{(m)}(x) D_{ij}, \quad m = 1, 2, \dots,$$

with coefficients $a^{ij}_{(m)}$ verifying (5.2), with VMO-moduli $\gamma_{a^{ij}_{(m)}}$ and L^∞-norms uniformly bounded by these of a^{ij}, and a sequence of functions

$$u^{(m)} \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega),$$

satisfying

$$\|u^{(m)}\|_{L^p(\Omega)} = 1, \quad \lim_{m\to\infty} \|\mathcal{L}^{(m)} u^{(m)}\|_{L^p(\Omega)} = 0.$$

By the Sarason characterization of VMO[4], for any fixed ball $B \subset \mathbb{R}^n$ the sequence $\left\{a^{ij}_{(m)} - (a^{ij}_{(m)})_B\right\}_{m\in\mathbb{N}}$ is compact in $L^1(B)$, and therefore there exists a subsequence converging a.e. in B. Considering an increasing sequence of balls, we can ensure $\lim_{m\to\infty} a^{ij}_{(m)} = \alpha^{ij}$ a.e. in Ω, with α^{ij} verifying (5.2).

Let

$$\mathcal{L}^{(\alpha)} \equiv \sum_{i,j=1}^{n} \alpha^{ij}(x) D_{ij}.$$

Since $\{u^{(m)}\}_{m\in\mathbb{N}}$ is equibounded in $W^{2,p}(\Omega)$ by (5.9), there exists a subsequence $\{u^{(m')}\}_{m\in\mathbb{N}}$ *weakly* converging to $u^{(\alpha)} \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$ as $m' \to \infty$ and such that

$$\lim_{m'\to\infty} \|u^{(m')}\|_{L^p(\Omega)} = \|u^{(\alpha)}\|_{L^p(\Omega)}.$$

Further on, for each $\varphi \in L^{p'}(\Omega)$, $1/p + 1/p' = 1$, we have

$$\begin{aligned}
&\int_\Omega \left|\left(\mathcal{L}^{(m')} u^{(m')} - \mathcal{L}^{(\alpha)} u^{(\alpha)}\right)\varphi\right| \\
&\quad \le \int_\Omega \left|\mathcal{L}^{(m')} u^{(m')} - \mathcal{L}^{(m')} u^{(\alpha)}\right||\varphi| + \int_\Omega \left|\mathcal{L}^{(m')} u^{(\alpha)} - \mathcal{L}^{(\alpha)} u^{(\alpha)}\right||\varphi| \\
&\quad \le \lambda \int_\Omega \left|D_{ij} u^{(m')} - D_{ij} u^{(\alpha)}\right||\varphi| + \int_\Omega \left|a^{ij}_{(m')} - \alpha^{ij}\right|\left|D_{ij} u^{(\alpha)}\right||\varphi|,
\end{aligned}$$

and the right-hand side tends to 0 as $m' \to \infty$ as result of the weak $W^{2,p}$-convergence of $u^{(m')}$ to $u^{(\alpha)}$ and the almost everywhere convergence of $a^{ij}_{(m')}$ to α^{ij}. As consequence, $\mathcal{L}^{(m')} u^{(m')}$ converges weakly in $L^p(\Omega)$ to $\mathcal{L}^{(\alpha)} u^{(\alpha)}$ and

$$\|\mathcal{L}^{(\alpha)} u^{(\alpha)}\|_{L^p(\Omega)} = \lim_{m'\to\infty} \|\mathcal{L}^{(m')} u^{(m')}\|_{L^p(\Omega)} = 0.$$

Hence $\mathcal{L}^{(\alpha)} u^{(\alpha)} = 0$ a.e. in Ω and the uniqueness result, Theorem 5.13 yields $u^{(\alpha)} = 0$, that is $\lim_{m'\to\infty} \|u^{(m')}\|_{L^p(\Omega)} = 0$, which contradicts $\|u^{(m)}\|_{L^p(\Omega)} = 1$. □

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[4] $f \in \mathrm{VMO} \Rightarrow \lim_{y\to 0} \|f(\cdot - y) - f(\cdot)\|_* = 0 \Rightarrow \|f(\cdot - y) - f(\cdot)\|_* \le C\gamma_f$. Therefore the usual molifiers converge to f in BMO, that is, there exist $f^{(m)} \in C^\infty$ such that $\lim_{m\to\infty} f^{(m)} = f$ in BMO and $\gamma_{f^{(m)}} \le \gamma_f$.

We are in a position now to derive strong $W^{2,p}$-solvability of the Dirichlet problem (5.8) for each $p \in (1, \infty)$.

Theorem 5.15 (**Strong solvability of** (5.8))**.** *Under the assumptions* (5.2) *and* $\partial\Omega$, *the Dirichlet problem* (5.8) *admits a* unique strong solution $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ *for each* $f \in L^p(\Omega)$ *with* $p \in (1, \infty)$.

Proof. If $u_1, u_2 \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ are two solutions of (5.8) with a given right-hand side f, their difference $u_1 - u_2$ solves a *homogeneous* Dirichlet problem whence $u_1 \equiv u_2$ by Theorem 5.13.

The existence of a strong solution can be obtained in a standard way, once having (5.10). In fact, it suffices to approximate a^{ij}'s by smooth coefficients, apply the Calderón–Zygmund L^p-theory of elliptic equations with continuous coefficients, and use (5.10) to get convergence of the approximating Dirichlet problems to (5.8) as did in the proof of Theorem 5.14.

Alternatively, strong solvability of (5.8) can be proved by the *method of continuity*[5], including (5.8) in one-parameter family of problems

$$\begin{cases} \mathcal{L}_s u \equiv s\mathcal{L}u + (1-s)\Delta u = f(x) \quad \text{a.e. } \Omega, \\ u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega). \end{cases} \tag{5.11}$$

with $s \in [0, 1]$.

It is clear that $\mathcal{L}_s$ is a uniformly elliptic operator with ellipticity constant depending only on λ, and the coefficients of $\mathcal{L}$ are $\mathrm{VMO} \cap L^\infty$-functions independently of s. Further on, $\mathcal{L}_s$ may be considered as a bounded linear operator from the Banach space $\mathfrak{B} = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ into $L^p(\Omega)$, and the solvability of the Dirichlet problem (5.11) for arbitrary $f \in L^p(\Omega)$ is equivalent of the *invertibility* of the mapping $\mathcal{L}_s$. Let u_s be a solution to (5.11). It follows from (5.10) that

$$\|u_s\|_{\mathfrak{B}} \le C \big\|\mathcal{L}_s u_s\big\|_{L^p(\Omega)}$$

with a constant C *independent* of $s \in [0, 1]$. Since, by the Calderón–Zygmund L^p-theory, $\mathcal{L}_0 \equiv \Delta$ maps $\mathfrak{B}$ *onto* $L^p(\Omega)$, the *method of continuity* ensures that also $\mathcal{L}_1 \equiv \mathcal{L}\colon\ \mathfrak{B} \to L^p(\Omega)$ is *onto* which means strong solvability of (5.11) with $s = 1$, that is, of (5.8). □

[5]Let $\mathfrak{B}$ be a Banach space, $\mathfrak{N}$ a normed linear space, and let $\mathcal{L}_0, \mathcal{L}_1\colon\ \mathfrak{B} \to \mathfrak{N}$ be bounded linear operators. For each $s \in [0, 1]$ set

$$\mathcal{L}_s := s\mathcal{L}_1 + (1-s)\mathcal{L}_0$$

and suppose that there exists a constant C, independent of s, such that

$$\|x\|_{\mathfrak{B}} \le C\|\mathcal{L}_s x\|_{\mathfrak{N}} \quad \forall s \in [0, 1].$$

Then $\mathcal{L}_1$ maps $\mathfrak{B}$ *onto* $\mathfrak{N}$ *if and only if* $\mathcal{L}_0$ maps $\mathfrak{B}$ *onto* $\mathfrak{N}$ ([7, Theorem 5.2]).

6. Concluding Remarks

6.1. General linear elliptic operators. The results of Section 5 can be easily extended to general linear elliptic operators

$$\mathcal{L} \equiv \sum_{i,j=1}^{n} a^{ij}(x)D_{ij} + \sum_{i=1}^{n} b^i(x)D_i + c(x)$$

with VMO $\cap\, L^\infty$ principal coefficients a^{ij}, and with b^i and c taken in suitable Lebesgue spaces to ensure $b^iD_iu, cu \in L^p$ whenever $u \in W^{2,p}$. For the validity of Theorems 5.13, 5.14 and 5.15, $c(x) \le 0$ for a.a. $x \in \Omega$ must be required as well.

Moreover, relevant regularity theories can be developed in another functional spaces such as Sobolev–Morrey, Sobolev–Campanato, weighted L^p, etc.

6.2. Oblique derivative problem. A $W^{2,p}$-theory, similar to the one presented above, can be developed (see [9, Chapter 2]) for the *oblique derivative problem*

$$\begin{cases} \mathcal{L}u = f(x) & \text{a.e. } \Omega, \\ \dfrac{\partial u}{\partial \boldsymbol{\ell}} + \sigma(x)u = \varphi & \text{in the sense of trace on } \partial\Omega, \end{cases}$$

where $\boldsymbol{\ell}\colon \partial\Omega \to \mathbb{R}^n$ is a prescribed unit vector field on $\partial\Omega$ which points *strictly* outwards Ω, $\boldsymbol{\ell}\cdot\boldsymbol{\nu} > 0$ on $\partial\Omega$ with $\boldsymbol{\nu}$ being the outward normal to $\partial\Omega$.

6.3. Second-order linear parabolic operators. Without essential difficulties, a relevant $W^{2,p}$-theory can be developed (cf. [9, Chapter 2]) also for linear parabolic operators with VMO-principal coefficients

$$\partial_t - \sum_{i,j=1}^{n} a^{ij}(x,t)D_{x_ix_j}.$$

Now the Euclidean norm $|x| = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$ must be replaced by the *parabolic* one $\rho(x,t) = \max\{|x|, \sqrt{t}\}$ and the fundamental solution to use in the corresponding representation formulae is that of a parabolic operator

$$\Gamma(x,t;y,s) = \begin{cases} \dfrac{(4\pi s)^{-n/2}}{\sqrt{\det\{a^{ij}(x,t)\}}} \exp\left\{ -\dfrac{\sum_{i,j=1}^n A^{ij}(x,t)y_iy_j}{4s} \right\} & \text{if } s > 0, \\ 0 & \text{if } s \le 0 \end{cases}$$

with $\left\{A^{ij}(x,t)\right\}$ being the inverse matrix $\left\{a^{ij}(x,t)\right\}^{-1}$.

6.4. Higher order equations and systems with VMO principal coefficients. The technique presented in Section 5 can be successfully applied to develop L^p-regularity theory for:

- **$2m$-order elliptic operators**

$$\sum_{|\beta|=2m} a^\beta(x)D^\beta$$

with $a^\beta \in \mathrm{VMO} \cap L^\infty$, and where ellipticity means existence of a constant $\lambda > 0$ such that

$$\lambda^{-1}|\xi|^{2m} \leq \sum_{|\beta|=2m} a^\beta(x)\xi^\beta \leq \lambda|\xi|^{2m}.$$

Here the fundamental solution of the operators has the form $\Gamma(x;y) = |y|^{2m-n}P(x;\bar{y})$, $\bar{y} = y/|y|$, where $P(x,\bar{y})$ is a real analytic function with respect to $\bar{y}$;

- **Linear elliptic systems**

$$\sum_{|\beta|=2b} \mathbf{A}_\beta(x)D^\beta \mathbf{u} = \mathbf{f}(x)$$

where $\mathbf{u}\colon\ \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$, $\mathbf{f}\colon\ \Omega \to \mathbb{R}^m$ and $\mathbf{A}_\beta = \left\{a^{ij}_\beta(x)\right\}_{i,j=1}^m \in \mathbb{R}^{m\times n}$ with $a^{ij}_\beta \in \mathrm{VMO} \cap L^\infty$ satisfy the ellipticity condition

$$\det\left\{\sum_{|\beta|=2b} \mathbf{A}_\beta(x)\xi^\beta\right\} \geq \lambda|\xi|^{2bm};$$

- **Linear parabolic systems**

$$\partial_t \mathbf{u}(x,t) - \sum_{|\beta|=2b} \mathbf{A}_\beta(x,t)D^\beta \mathbf{u} = \mathbf{f}(x,t)$$

with $\mathrm{VMO} \cap L^\infty$-coefficients, where parabolicity in the sense of Petrovskii is assumed, that is, the h-roots of the m-degree polynomial

$$\det\left\{h\mathrm{Id}_{\mathbb{R}^m} - \sum_{|\beta|=2b} \mathbf{A}_\beta(x,t)(i\xi)^\beta\right\} = 0$$

satisfy

$$\operatorname{Re} h_s(x,t,\xi) \leq -\lambda|\xi|^{2b} \quad \lambda > 0,\ s = 1,\dots,m.$$

6.5. Singular integral operators in homogeneous spaces. The results from Section 4 about boundedness of singular integral operators and their commutators can be extended to the settings of *homogeneous spaces* (X,d,μ). Here X is a set endowed with a *quasidistance* $d\colon\ X\times X \to [0,\infty)$ satisfying

i) $d(x,y) = 0 \Longleftrightarrow x = y$;
ii) $d(x,y) = d(y,x)$;
iii) $d(x,y) \leq C_d\big(d(x,z) + d(z,y)\big)$,

and such that the balls with respect to d form a base for a complete system of neighbourhoods of X, so that X is a Hausdorff space, and μ is a Borel measure which supports the *doubling condition*

$$\mu\big(B_{2r}\big) \leq C_\mu \mu\big(B_r\big).$$

For example, let $\alpha_i \geq 1$, $\alpha = \sum_{i=1}^n \alpha_i$ and consider the function

$$F(x,\rho) = \frac{x_1^2}{\rho^{2\alpha_1}} + \frac{x_2^2}{\rho^{2\alpha_2}} + \cdots + \frac{x_n^2}{\rho^{2\alpha_n}}.$$

It is decreasing with respect to ρ and therefore there exists a unique solution $\rho(x)$ of the equation $F(x,\rho) = 1$.

It turns out that $\big(\mathbb{R}^n, \rho, dx\big)$ is a homogeneous space and the balls with respect to ρ are the *ellipsoids*

$$\left\{x \in \mathbb{R}^n\colon \quad \frac{x_1^2}{\rho^{2\alpha_1}} + \frac{x_2^2}{\rho^{2\alpha_2}} + \cdots + \frac{x_n^2}{\rho^{2\alpha_n}} < 1\right\},$$

which are dilations of the n-dimensional Euclidean unit ball with respect to ρ.

Employing the machinery described in Section 4, boundedness of singular integrals and commutators with kernels of mixed homogeneity

$$k\big(t^{\alpha_1}x_1, t^{\alpha_2}x_2, \ldots, t^{\alpha_n}x_n\big) = t^{-\alpha}k(x)$$

can be obtained, and these serve to develop L^p-theory of *anisotropic* elliptic PDEs.

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