Universität des Saarlandes
Fachrichtung 6.1 – Mathematik
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Calculus of Variations (Summer Term 2016)
Exam-Part W

Problem E.1 (10 Points)

Find the coordinates of the point(s) nearest the origin on the surface $xyz = R^3$, for $x, y, z \geq 0$.

Show (using the transversal conditions and the Euler-Lagrange equations) that if we were to draw a line between this point and the origin, it would be a transversal of minimum length between the origin and the surface.

Problem E.2 (10 Points)

Solve the isoperimetric problem inside a square region, i.e., what is the shape that contains the largest area without exceeding a given perimeter L, given that the shape must be entirely contained in a square with sides $2W$ in length.

Note that the problem is uninteresting for $W > \dfrac{L}{2\pi}$ because a circle of radius $R = \dfrac{L}{2\pi}$ satisfies the isoperimetric constraint, and fits inside the square, and this is clearly the maximal area region (though there are actually multiple possible circles that might fit).

Likewise, $8W < L$ is uninteresting, because we cannot meet the perimeter constraint without having a concave shape, so the obvious solution is to contain the entire area of the square, but have the perimeter dip into the shape along a line enclosing zero area.

So, we consider the case
$$\frac{L}{8} < W < \frac{L}{2\pi}.$$

(*Hint:* Think about symmetry of the problem)

Problem E.3 (10 Points)

Consider the functional

$$J[y] = \int_0^2 \left[(y'^2 + y^2)(y - x)^2 - \frac{4}{3}y^3 + 2xy^2 \right] dx$$

$$y(0) = 0, \qquad y(2) = e$$

a) Find the corresponding Euler-Lagrange equation and find out at what points in $[0, 2]$ it must hold.

b) Show that $y = x$ and $y = \alpha e^x$ are solutions of the equation found in part a).

c) Find all possible corner points for the extremals of J.

d) Find a possible extremal for our problem.

Problem E.4 (10 Points)

Use Ritz's method to find an approximate solution to minimize the

$$J[y] = \int_0^2 \left[2xy + y^2 + y'^2 \right] dx,$$

where $y(0) = y(2) = 0$. Compare your solution to one found directly from the Euler-Lagrange equations.

Problem E.5 (10 Points)

Solve the following optimal control problems with the help of PMP:

(a) Minimize

$$F(x) = \int_0^{10} x^2 dt$$

subject to

$$|\ddot{x}| \leqslant 1, \text{and } x(0) = 1.$$

(b) Minimize T subject to

$$\int_0^T \ddot{x}^2 dt = 4$$

and

$$x(0) = 1, \text{and } \dot{x}(0) = 1, \text{and } \dot{x}(T) = -2.$$

Deadline for submission: Friday, September 30, 2016