Calculus of Variations
Summer Term 2016

Lecture 11

Universität des Saarlandes

14. Juni 2016

Purpose of Lesson:

- To finish a prove of our general result about problems with inequality constraints
- To introduce the notion of a broken extremal
- To discuss the properties of broken extremals

Recall formula (10.3) from Lecture 10:

$$\boxed{\left\{g' F_{y'} + F - y' F_{y'}\right\}\Big|_{x=x^*} = 0}. \tag{10.3}$$

- From (10.3) it follows that we may write the condition in x^* in terms of limits from the left and right, e.g.

$$\left[g' F_{y'} + F - y' F_{y'}\right]_{x^*-} - \left[g' F_{y'} + F - y' F_{y'}\right]_{x^*+} = 0$$

- Taking into account that $y' = g'$ on the RHS of x^* we get

$$\begin{aligned} 0 &= \left[g' F_{y'} + F - y' F_{y'}\right]_{x^*-} - \left[g' F_{y'} + F - g' F_{y'}\right]_{x^*+} \\ &= \left[(g' - y') F_{y'} + F\right]_{x^*-} - F\big|_{x^*+} \end{aligned}$$

or

$$\left[(g' - y') F_{y'}\right]_{x^*-} = F\big|_{x^*+} - F\big|_{x^*-}. \qquad (11.1)$$

- Consider the term $\left\{ F\big|_{x^{*+}} - F\big|_{x^{*-}} \right\}$.
- Note that at the "join" $y(x^*) = g(x^*)$, so if the two limits of F differ it is because of a difference in y' on either side of the join.
- Treat F as a function of just y', i.e.,

$$F(x, y, y') = q_{x,y}(y') = q(y').$$

- Taking $q(y') = F(x, y, y')$ we get

$$\frac{d}{dz} q(z) = \frac{\partial F}{\partial y'}(x, y, y')\bigg|_{y'=z}.$$

So

$$q'(c) = \frac{\partial F}{\partial y'}(x^*, y^*, c).$$

- Hence

$$\begin{aligned} F\Big|_{x^*+} - F\Big|_{x^*-} &= q(g'(x^*)) - q(y'(x^*)) \\ &= \left[g'(x^*) - y'(x^*)\right] q'(c) \\ &= \left[g'(x^*) - y'(x^*)\right] \frac{\partial F}{\partial y'}(x^*, y^*, c) \end{aligned}$$

- So, the condition (11.1) can be rewritten as follows

$$\left[(g' - y')\frac{\partial F}{\partial y'}\right]_{x^*-} = \left[g'(x^*) - y'(x^*)\right] \frac{\partial F}{\partial y'}(x^*, y^*, c)$$

- Hence

$$\left[(g' - y')\left(\frac{\partial F}{\partial y'}(x, y, y') - \frac{\partial F}{\partial y'}(x, y, c)\right)\right]_{x=x^*} = 0$$

for some c between $g'(x^*)$ and $y'(x^*)$.

$$(g'(x^*) - y'(x^*)) \left(\frac{\partial F}{\partial y'}(x^*, y(x^*), y'(x^*)) - \frac{\partial F}{\partial y'}(x^*, y(x^*), c) \right) = 0$$

So, there are two possibilities

- $g'(x^*) = y'(x^*)$, which means that y meets the boundary at a tangent to the boundary.
- $F_{y'}(x, y, y') - F_{y'}(x, y, c) = 0$. This latter condition holds when $F_{y'}$ is constant with respect to y', i.e.,

$$\frac{\partial^2 F}{\partial y'^2} = 0.$$

Remark

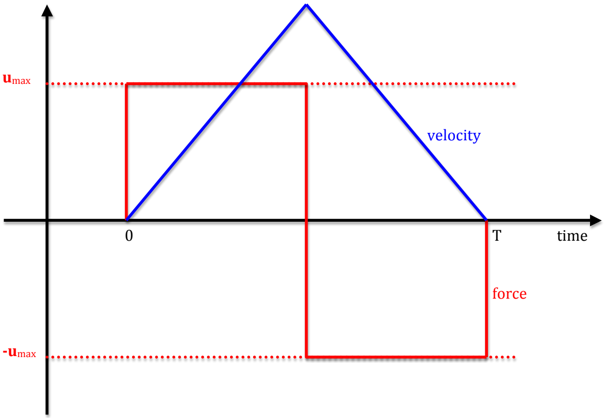
In the lake example, $F_{y'y'} \neq 0$.

Example 11.1 : parking a car (see Example 10.1)

- Revisit the problem of parking a car.
- If we think about the problem, it makes no sense unless there is maximum force u_{max}.
  - Otherwise we move from A to B arbitrarily fast.
- There are no valid E-L equation solutions.
- We must end-up in the boundary domain, e.g. $u = \pm u_{max}$.
  - Obvious solution is to accelerate as fast as possible until we get half-way, and then to decelerate as fast as possible.
  - $\dfrac{\partial F}{\partial \dot{u}} = 0$, so we don't have to stress about continuity (u is not continuous either).

Example 11.1: parking a car (cont.)

- Our solution is in the boundary domain, e.g. $u = \pm u_{max}$



- called a bang-bang controller.

§9. Broken extremals

- Until now we mostly stude the extremals curves with at least two well-defined derivatives.
- Obviously this is not always true.
- Broken extremals are continuous extremals for which the gradient has a discontinuty at one or more points.
- If a variational problem has a smooth extremal (That therefore satisfies the Euler-Lagrange equations), this will be better than a broken one.
- But some problems don't admit smooth extremals.

Example 11.2

Find $y(x)$ to minimize

$$J[y] = \int_{-1}^{1} y^2(1 - y')^2 dx$$

subject to $y(-1) = 0$ and $y(1) = 1$.

Example 11.2 (cont.)

- There is no explicit x dependence inside the integral, so we can find

$$\begin{aligned} F - y' F_{y'} &= c_1 = const \\ y^2(1-y')^2 + 2y'y^2(1-y') &= c_1 \\ y^2(1-y')\left[1+y'\right] &= c_1 \\ y^2\left[1-y'^2\right] &= c_1 \end{aligned}$$

- If $c_1 = 0$ we get the singular solutions

$$y = 0 \qquad \text{or} \qquad y = \pm x + B.$$

Neither of these satisfies both end-ponts conditions $y(-1) = 0$ and $y(1) = 1$, so $c_1 \neq 0$ (we think).

Example 11.2 (cont.)

- Given $c_1 \neq 0$

$$
\begin{aligned}
y^2\left[1 - y'^2\right] &= c_1 \\
y'^2 &= \frac{y^2 - c_1}{y^2} \\
\frac{dy}{dx} &= \pm\frac{1}{y}\sqrt{y^2 - c_1} \\
dx &= \pm\frac{y}{\sqrt{y^2 - c_1}}dy \\
x &= \pm\sqrt{y^2 - c_1} + c_2 \\
(x - c_2)^2 &= y^2 - c_1
\end{aligned}
$$

- The solution is a rectangular hyperbola.

Example 11.2 (cont.)

- Using the end-points conditions we find c_1 and c_2 from

$$(x - c_2)^2 = y^2 - c_1 .$$

$$\begin{aligned} y(-1) = 0 \quad &\Rightarrow \quad (-1 - c_2)^2 = -c_1 \\ y(1) = 1 \quad &\Rightarrow \quad (1 - c_2)^2 = 1 - c_1 \end{aligned}$$

- Addition of these two equations gives

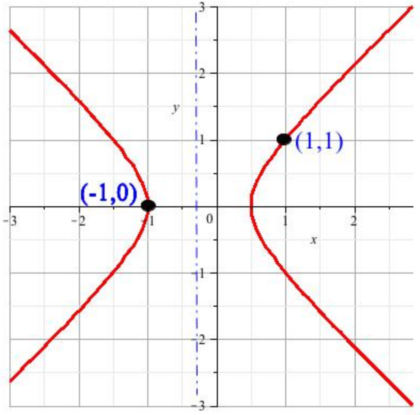
$$(1 - c_2)^2 = 1 + (1 + c_2)^2$$

which has solution $c_2 = -1/4$, and so $c_1 = -9/16$

$$y^2 = (x + 1/4)^2 - 9/16.$$

Example 11.2 (cont.)

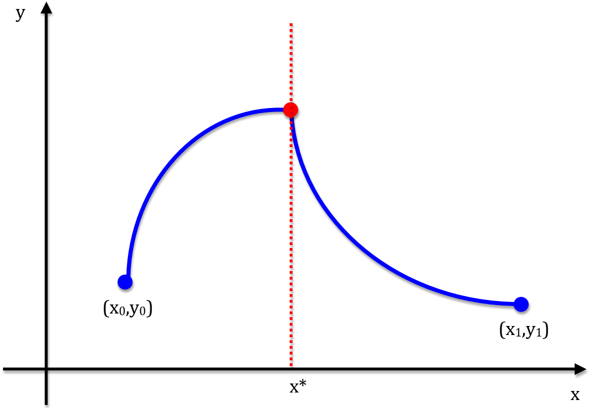
- The end-points are on opposite branches of the hyperbola!



- There is NO smooth extremal curve that connects $(-1, 0)$ and $(1, 1)$.

- Sometimes there is no smooth extremal.
- We must seek a broken extremal.
- Still want a continuous extremal.
- What should we do?
  - Previous smothness results suggest that we should use a smooth extremal when we can, and so we will try to minimize the number of corners.
  - We'll start by looking for curves with one corner.
  - But can we apply the Euler-Lagrange equations?

- If we have an extremal like this, can we use the Euler-Lagrange equations?



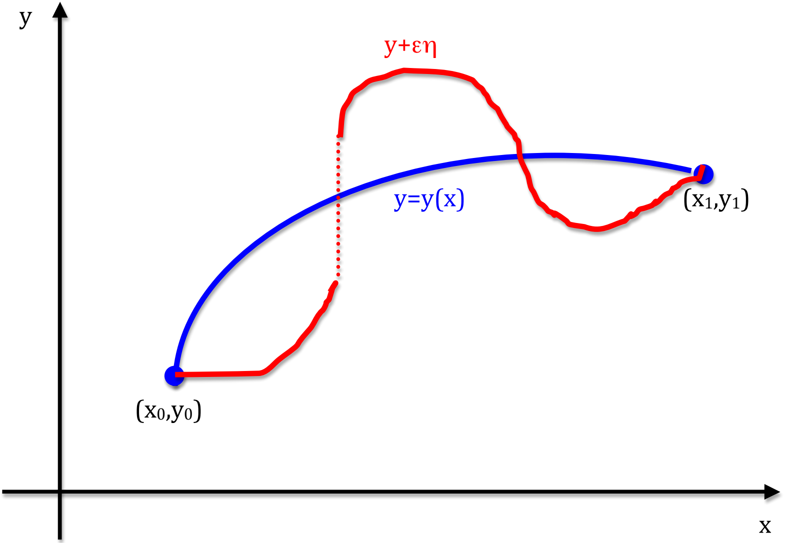
Smoothness theorem

Theorem 11.1

If the smooth curve $y(x)$ gives an extremal of a functional $J[y]$ over the class of all admissible curves in some ε neighborhood of y, then $y(x)$ also gives an extremal of a functional $J[y]$ over the class of all piecewise smooth curves in the same neighborhood.

Meaning:

We can extend our results to piecewise smooth curves (where a smooth result exists), not just curves with 2 continuous derivatives.


y
y+εη
y=y(x)
(x1,y1)
(x0,y0)
x

Proof Sketch

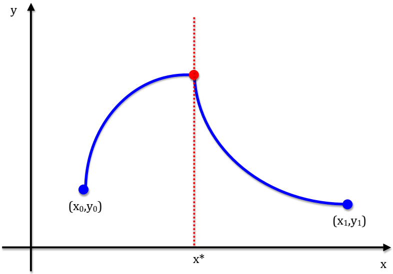
- The theorem assumes that there exists a smooth extremal (in this case a minimum for the purpose of illustration) y. Then for any other smooth curve $\hat{y} \in B_\varepsilon(y)$ we know $J[\hat{y}] > J[y]$.
- Assume for the moment that for a piecewise smooth function $\tilde{y} \in B_\varepsilon(y)$ we have $J[\tilde{y}] < J[y]$. We can approximate $\tilde{y}$ by a smooth curve $\hat{y}_\delta \in B_\varepsilon(y)$ by rounding off the edges of the discontinuity.
- Given that we can approximate the curve $\tilde{y}$ arbitrarily closely by a smooth curve $\hat{y}_\delta$, for which we already know $J[\hat{y}_\delta] > J[y]$. We get a contradiction with $J[\tilde{y}] < J[y]$, and so no such alternative extremal can exist.

So What We Do?

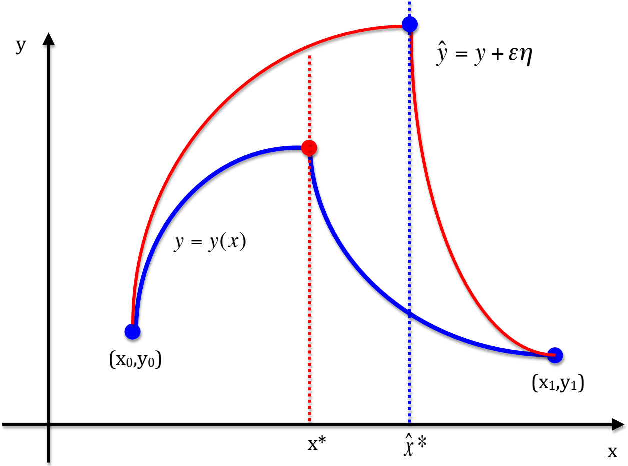
- Break the functional into two parts:

$$J[y] = J_1[y] + J_2[y] = \int\limits_{x_0}^{x^*} F(x, y_1, y_1')dx + \int\limits_{x^*}^{x_1} F(x, y_2, y_2')dx$$

- We require y to have two continuous derivatives everywhere except at x^*, and $y_1(x^*) = y_2(x^*)$.



Possible Perturbations:



The location of the ”corner” can also be perturbed

The First Variation: part 1

- We get the first component of the first variation by considering a problem with only one fixed end-point, and allowing x^* to vary, so that

$$0 = \frac{d\phi_1(\varepsilon)}{d\varepsilon}\bigg|_{\varepsilon=0} = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \int\limits_{x_0}^{\hat{x}^*} F(x, y_1 + \varepsilon\eta, y_1' + \varepsilon\eta')dx$$

$$= \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \int\limits_{x_0}^{x^*+\varepsilon X} F(x, y_1 + \varepsilon\eta, y_1' + \varepsilon\eta')dx$$

$$= XF(x, y_1, y_1')\bigg|_{x=x^*} + F_{y_1'}\eta\bigg|_{x=x^*} + \int\limits_{x_0}^{x^*} \left(F_{y_1} - \frac{d}{dx}F_{y_1'}\right)\eta dx$$

The First Variation: part 1

- The perturbed point $(\hat{x}^*, \hat{y}^*)$ and perturbed function η must satisfy certain conditions to be compatible.
- Remember that

$$\begin{aligned} \hat{x}^* &= x^* + \varepsilon X \\ \hat{y}^* &= y^* + \varepsilon Y \end{aligned}$$

- Notice that

$$\hat{y}^* = y(x^* + \varepsilon X) + \varepsilon\eta(x^* + \varepsilon X).$$

- From Taylor's theorem, for small ε

$$\begin{aligned} y(x^* + \varepsilon X) &= y(x^*) + \varepsilon X y'(x^*) + O(\varepsilon^2) \\ &= y^* + \varepsilon X y'(x^*) + O(\varepsilon^2) \\ \varepsilon\eta(x^* + \varepsilon X) &= \varepsilon\eta(x^*) + O(\varepsilon^2) \end{aligned}$$

The First Variation: part 1

- So

$$\begin{aligned} y^* + \varepsilon Y &= y^* + \varepsilon X y'(x^*) + \varepsilon \eta(x^*) + O(\varepsilon^2) \\ \varepsilon Y &= \varepsilon X y'(x^*) + \varepsilon \eta(x^*) + O(\varepsilon^2) \\ \eta(x^*) &= Y - X y'(x^*) + O(\varepsilon) \end{aligned}$$

- Thus, we have

$$\boxed{\eta(x^*) = Y - X y'(x^*) + O(\varepsilon)} \tag{11.2}$$

The First Variation: part 1

- Substituting the compatibility constraint (11.2) into the our first variation we get

$$0 = \Big[XF + F_{y_1'}\eta\Big]_{x=x^*} + \int\limits_{x_0}^{x^*} \left(F_{y_1} - \frac{d}{dx}F_{y_1'}\right)\eta dx$$

$$= XF\Big|_{x=x^*} + \left[Y - Xy_1'(x^*)\right] F_{y_1'}\Big|_{x=x^*} + \int\limits_{x_0}^{x^*} \left(F_{y_1} - \frac{d}{dx}F_{y_1'}\right)\eta dx$$

$$= X\Big[F - y_1' F_{y_1'}\Big]_{x=x^*} + Y F_{y_1'}\Big|_{x=x^*} + \int\limits_{x_0}^{x^*} \left(F_{y_1} - \frac{d}{dx}F_{y_1'}\right)\eta dx$$

The First Variation: part 1

- So, we get an integral term which results in the E-L equation, plus the additional constraint

$$X\left[F - y_1' F_{y_1'}\right]_{x=x^*} + Y F_{y_1'}\Big|_{x=x^*} = 0 \qquad (11.3)$$

The First Variation: part 2

- Note that, for the second component of the First Variation we get a similar extra term, e.g.

$$-X\left[F - y_2' F_{y_2'}\right]_{x=x^*} - Y F_{y_2'}\Big|_{x=x^*} = 0. \qquad (11.4)$$

- The sign is reversed because it corresponds to the x_0 term (as opposed to the x_1 term for δJ_1).

- The combined First Variation (minus the terms that result from the Euler-Lagrange equation which must be zero) is

$$X\left[F - y_1' F_{y_1'}\right]_{x=x^*} + Y F_{y_1'}\Big|_{x=x^*} - X\left[F - y_2' F_{y_2'}\right]_{x=x^*} - Y F_{y_2'}\Big|_{x=x^*} = 0.$$

Conditions

- We rearrange to give

$$0 = X\left\{\left[F(x,y_1,y_1') - y_1' F_{y_1'}\right] - \left[F(x,y_2,y_2') - y_2' F_{y_2'}\right]\right\}_{x=x^*} + Y\left\{F_{y_1'} - F_{y_2'}\right\}_{x=x^*}.$$

- Note that the point of discontinuity may vary freely, so we may independently vary X and Y or set one or both to zero. Hence, we can separate the condition to get two conditions

$$\left[F(x,y_1,y_1') - y_1' F_{y_1'} - F(x,y_2,y_2') + y_2' F_{y_2'}\right]_{x=x^*} = 0$$

$$\left\{F_{y_1'} - F_{y_2'}\right\}_{x=x^*} = 0$$

Weierstrass-Erdman

- We can write the conditions as

$$\left[F(x, y_1, y_1') - y_1' F_{y_1'}\right]_{x=x^*} = \left[F(x, y_2, y_2') - y_2' F_{y_2'}\right]_{x=x^*}$$

$$F_{y_1'}\Big|_{x=x^*} = F_{y_2'}\Big|_{x=x^*}$$

Called the Weierstrass-Erdmann Corner Conditions.

- Rather than separating y into y_1 and y_2 we may write the corner conditions in terms of limits from the left and right, e.g.

$$\left[F - y' F_{y'}\right]_{x=x^{*-}} = \left[F - y' F_{y'}\right]_{x=x^{*+}}$$

$$F_{y'}\Big|_{x=x^{*-}} = F_{y'}\Big|_{x=x^{*+}}$$

Solution

So the broken extremal solution must satisfy

- The Euler-Lagrange equations
- The Weierstrass-Erdmann Corner Conditions

$$\left[F - y' F_{y'}\right]_{x=x^{*-}} = \left[F - y' F_{y'}\right]_{x=x^{*+}}$$
$$F_{y'}\Big|_{x=x^{*-}} = F_{y'}\Big|_{x=x^{*+}}$$

must hold at any ”corner”.

Example 11.2 (cont. ii)

Find $y(x)$ to minimize

$$J[y] = \int\limits_{-1}^{1} y^2(1 - y')^2 dx$$

subject to $y(-1) = 0$ and $y(1) = 1$.

Example 11.2 (cont. ii)

- In the example considered

$$F - y' F_{y'} = y^2 \left(1 - y'^2\right)$$
$$F_{y'} = -2y^2 \left(1 - y'\right)$$

- Remember that $y = 0$ and $y = x + A$ are valid solutions to the Euler-Lagrange equations, and that for both of these solutions

$$F_{y'} = F - y' F_{y'} = 0,$$

so we can put a ”corner” where needed.

Example 11.2 (cont. ii)

- The solution must also satisfy the end-point conditions, so $y(-1) = 0$ and $y(1) = 1$, and therefore, as valid solution has $x^* = 0$ and

$$
\begin{aligned}
y_1 &= 0 \quad \text{for} \quad x \in [-1, x^*] \\
y_2 &= x \quad \text{for} \quad x \in [x^*, 1]
\end{aligned}
$$