

Calculus of Variations

Summer Term 2016

Lecture 12

Universität des Saarlandes

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Purpose of Lesson:

- To discuss numerical solutions of the variational problems
- To introduce Euler's Finite Difference Method and Ritz's Method.

§10. Numerical Solutions

Numerical Solutions:

The Euler-Lagrange equations may be hard to solve.

Natural response is to find numerical methods.

- 1 Numerical solution of the Euler-Lagrange equations
 - We won't consider these here (see other courses)
- 2 Euler's finite difference method
- 3 Ritz (Rayleigh-Ritz)
 - In $2D$: Kantorovich's method

Euler's Finite Difference Method

- We can approximate our function (and hence the integral) onto a finite grid.
- In this case, the problem reduces to a standard multivariable maximization (or minimization) problem, and we find the solution by setting the derivatives to zero.
- In the limit as the grid gets finer, this approximates the Euler-Lagrange equations.

Numerical approximation of integrals:

- use an arbitrary set of mesh points

$$a = x_0 < x_1 < x_2 < \dots < x_n = b.$$

- approximate

$$y'(x_i) = \frac{y_{i+1} - y_i}{x_{i+1} - x_i} = \frac{\Delta y_i}{\Delta x_i}$$

- rectangle rule

$$J[y] = \int_a^b F(x, y, y') dx \simeq \sum_{i=0}^{n-1} F\left(x_i, y_i, \frac{\Delta y_i}{\Delta x_i}\right) \Delta x_i = \widehat{J}[\mathbf{y}]$$

$\widehat{J}[\cdot]$ is a function of the vector $\mathbf{y} = (y_0, y_1, y_2, \dots, y_n)$.

Finite Difference Method (FDM)

- Treat this as a maximization of a function of $n + 1$ variables, so that we require

$$\frac{\partial \tilde{J}}{\partial y_i} = 0$$

for all $i = 0, 1, 2, \dots, n$.

- Typically use uniform grid so

$$\Delta x_i = \Delta x = \frac{b - a}{n}.$$

Example 12.1

Find extremals for

$$J[y] = \int_0^1 \left[\frac{1}{2}y'^2 + \frac{1}{2}y^2 - y \right] dx$$

with $y(0) = 0$ and $y(1) = 0$.

The Euler-Lagrange equation $y'' - y = -1$.

Example 12.1 (direct solution)

- E-L equation: $y'' - y = -1$
- Solution to homogeneous equation $y'' - y = 0$ is given by $e^{\lambda x}$ giving characteristic equation

$$\lambda^2 - 1 = 0,$$

so $\lambda = \pm 1$

- Particular solution $y = 1$.
- Final solution is

$$y(x) = Ae^x + Be^{-x} + 1.$$

Example 12.1 (direct solution)

- The boundary conditions $y(0) = y(1) = 0$ constrain

$$A + B = -1$$

$$Ae + Be^{-1} = -1$$

so $A = \frac{1 - e}{e^2 - 1}$ and $B = \frac{e - e^2}{e^2 - 1}$.

- Then the exact solution to the extremal problem is

$$y(x) = \frac{1 - e}{e^2 - 1} e^x + \frac{e - e^2}{e^2 - 1} e^{-x} + 1.$$

Example 12.1 (Euler's FDM)

Find extremals for

$$J[y] = \int_0^1 \left[\frac{1}{2}y'^2 + \frac{1}{2}y^2 - y \right] dx$$

Euler's FDM:

- Take the grid $x_i = i/n$, for $i = 0, 1, \dots, n$ so
 - end points $y_0 = 0$ and $y_n = 0$
 - $\Delta x = 1/n$ and $\Delta y_i = y_{i+1} - y_i$.
- So
 - $y'_i = \Delta y_i / \Delta x = n(y_{i+1} - y_i)$
 - and

$$y_i'^2 = n^2 (y_i^2 - 2y_i y_{i+1} + y_{i+1}^2).$$

Example 12.1 (Euler's FDM)

Find extremals for

$$J[y] = \int_0^1 \left[\frac{1}{2} y'^2 + \frac{1}{2} y^2 - y \right] dx$$

Its FDM approximation is

$$\begin{aligned} \tilde{J}[y] &= \sum_{i=0}^{n-1} F(x_i, y_i, y'_i) dx \\ &= \sum_{i=0}^{n-1} \frac{1}{2} n^2 (y_i^2 - 2y_i y_{i+1} + y_{i+1}^2) \Delta x + (y_i^2/2 - y_i) \Delta x \\ &= \sum_{i=0}^{n-1} \frac{1}{2} n (y_i^2 - 2y_i y_{i+1} + y_{i+1}^2) + \frac{y_i^2/2 - y_i}{n}. \end{aligned}$$

Example 12.1 (end-conditions)

- We know the end conditions $y(0) = y(1) = 0$, which imply that

$$y_0 = y_n = 0.$$

- Include them into the objective using Lagrange multipliers

$$\mathcal{H}[\mathbf{y}] = \sum_{i=0}^{n-1} \frac{1}{2} n \left(y_i^2 - 2y_i y_{i+1} + y_{i+1}^2 \right) + \frac{y_i^2/2 - y_i}{n} + \lambda_0 y_0 + \lambda_n y_n.$$

Example 12.1 (Euler's FDM)

- Taking derivatives, note that y_i only appears in two terms of the FDM approximation

$$\mathcal{H}[\mathbf{y}] = \sum_{i=0}^{n-1} \frac{1}{2} n \left(y_i^2 - 2y_i y_{i+1} + y_{i+1}^2 \right) + \frac{y_i^2/2 - y_i}{n} + \lambda_0 y_0 + \lambda_n y_n$$

$$\frac{\partial \mathcal{H}[\mathbf{y}]}{\partial y_i} = \begin{cases} n(y_0 - y_1) + \frac{y_0 - 1}{n} + \lambda_0 & \text{for } i = 0 \\ n(2y_i - y_{i+1} - y_{i-1}) + y_i/n - 1/n & \text{for } i = 1, \dots, n-1 \\ n(y_n - y_{n-1}) + \lambda_n & \text{for } i = n \end{cases}$$

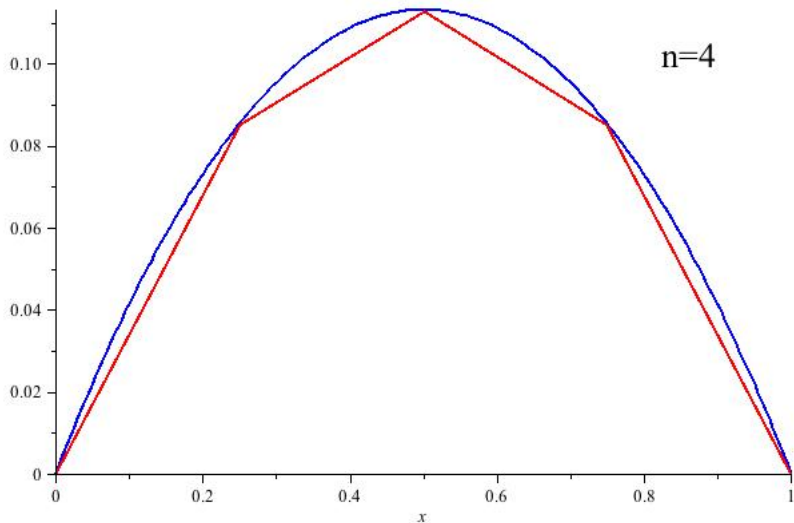
- We need to set the derivatives to all be zero, so we now have $n + 3$ linear equations, including $y_0 = y_n = 0$, and $n + 3$ variables including the two Lagrange multipliers.
- We can solve this system numerically using, e.g., Maple.

Example 12.1 (Euler's FDM)

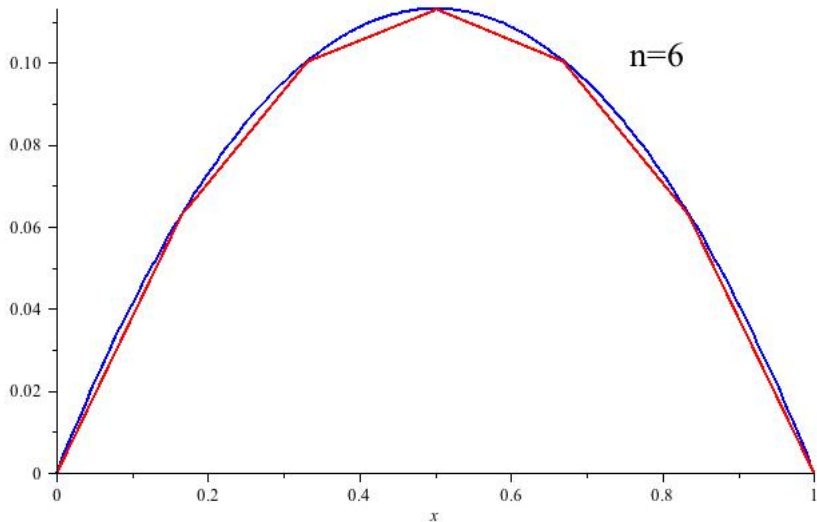
- First $n + 1$ terms of \mathbf{z} give \mathbf{y}
- Last two terms of \mathbf{z} give the Lagrange multipliers λ_0 and λ_n .
- Solving the system we get for $n = 4$

$$y_1 = y_3 = 0.08492201040, \quad y_2 = 0.1126516464$$

Example 12.1 (results)



Example 12.1 (results)



Convergence of Euler's FDM

$$\widehat{J}[\mathbf{y}] = \sum_{i=0}^{n-1} F \left(x_i, y_i, \frac{\Delta y_i}{\Delta x} \right) \Delta x \quad \text{and} \quad \Delta y_i = y_{i+1} - y_i$$

Only two terms in the sum involve y_i , so

$$\begin{aligned} \frac{\partial \widehat{J}}{\partial y_i} &= \frac{\partial}{\partial y_i} F \left(x_{i-1}, y_{i-1}, \frac{\Delta y_{i-1}}{\Delta x} \right) + \frac{\partial}{\partial y_i} F \left(x_i, y_i, \frac{\Delta y_i}{\Delta x} \right) \\ &= \frac{1}{\Delta x} \frac{\partial F}{\partial y'_i} \left(x_{i-1}, y_{i-1}, \frac{\Delta y_{i-1}}{\Delta x} \right) \\ &\quad + \frac{\partial F}{\partial y_i} \left(x_i, y_i, \frac{\Delta y_i}{\Delta x} \right) - \frac{1}{\Delta x} \frac{\partial F}{\partial y'_i} \left(x_i, y_i, \frac{\Delta y_i}{\Delta x} \right) \\ &= \frac{\partial F}{\partial y_i} (x_i, y_i, y'_i) - \frac{\frac{\partial F}{\partial y'_i} \left(x_i, y_i, \frac{\Delta y_i}{\Delta x} \right) - \frac{\partial F}{\partial y'_i} \left(x_{i-1}, y_{i-1}, \frac{\Delta y_{i-1}}{\Delta x} \right)}{\Delta x} \end{aligned}$$

Convergence of Euler's FDM

$$\frac{\partial \hat{J}}{\partial y_i} = \frac{\partial F}{\partial y_i}(x_i, y_i, y'_i) - \frac{\frac{\partial F}{\partial y'_i}(x_i, y_i, \frac{\Delta y_i}{\Delta x}) - \frac{\partial F}{\partial y'_i}(x_{i-1}, y_{i-1}, \frac{\Delta y_{i-1}}{\Delta x})}{\Delta x} = 0.$$

In limit $n \rightarrow \infty$, then $\Delta x \rightarrow 0$, and so we get

$$\frac{\partial F}{\partial y} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) = 0$$

which are the Euler-Lagrange equations.

- i.e., the finite difference solution converges to the solution of the Euler-Lagrange equations.

Remarks

- There are lots of ways to improve Euler's FDM
 - use a better method of numerical quadrature (integration)
 - trapezoidal rule
 - Simpson's rule
 - Romberg's method
 - use a non-uniform grid
 - make it finer where there is more variation
- We can use a different approach that can be even better.

Ritz's Method

- In Ritz's method (called Kantorovich's method where there is more than one independent variable), we approximate our functions (the extremal in particular) using a family of simple functions.
- Again we can reduce the problem into a standard multivariable maximization problem, but now we seek coefficients for our approximation.

Assume we can approximate $y(x)$ by

$$y(x) = \phi_0(x) + c_1\phi_1(x) + c_2\phi_2(x) + \cdots + c_n\phi_n(x)$$

where we choose a convenient set of functions $\phi_j(x)$ and find the values of c_j which produce an extremal.

For fixed end-points problem:

- Choose $\phi_0(x)$ to satisfy the end conditions.
- Then $\phi_j(x_0) = \phi_j(x_1) = 0$ for $j = 1, 2, \dots, n$

The ϕ can be chosen from standard sets of functions, e.g. power series, trigonometric functions, Bessel's functions, etc. (but must be linearly independent).

- Select $\{\phi_j\}_{j=0}^n$
- Approximate

$$y_n(x) = \phi_0(x) + c_1\phi_1(x) + c_2\phi_2(x) + \cdots + c_n\phi_n(x)$$

- Approximate $J[y] \simeq J[y_n] = \int_{x_0}^{x_1} F(x, y_n, y_n') dx$.
- Integrate to get $J[y_n] = J_n(c_1, c_2, \dots, c_n)$.
- J_n is a known function of n variables, so we can maximize (or minimize) it as usual by

$$\frac{\partial J_n}{\partial c_i} = 0$$

for all $i = 1, 2, \dots, n$.

- Assume the extremal of interest is a minimum, then for the extremal

$$J[y] < J[\hat{y}]$$

for all \hat{y} within the neighborhood of y .

- Assume our approximating function y_n is close enough to be in that neighborhood, then

$$J[y] \leq J[y_n] = J_n[\mathbf{c}]$$

so the approximation provides an **upper bound** on the minimum $J[y]$.

- Another way to think about it is that we optimize on a smaller set of possible functions y , so we can't get quite as good a minimum.

Application of the Ritz method:

Example 12.1a

Find extremals for

$$J[y] = \int_0^1 \left[\frac{1}{2}y'^2 + \frac{1}{2}y^2 - y \right] dx$$

with $y(0) = 0$ and $y(1) = 0$.

The Euler-Lagrange equation $y'' - y = 1$, but we shall bypass the Euler-Lagrange equation to use Ritz's method.

$$y_n(x) = \phi_0(x) + \sum_{i=1}^n c_i \phi_i(x)$$

where we take $\phi_0(x) = 0$ and $\phi_i(x) = x^i (1 - x)^i$.

Example 12.1a

- Simple approximation $y_1 = c_1\phi_1(x)$ we get

$$J_1[c_1] = J[y_1] = \int_0^1 \left[\frac{1}{2}c_1^2\phi_1'^2 + \frac{1}{2}c_1^2\phi_1^2 - c_1\phi_1 \right] dx.$$

- Now $\phi_1(x) = x(1-x)$ so $\phi_1' = 1-2x$, and

$$\begin{aligned} J_1[c_1] &= \int_0^1 \left[\frac{c_1^2}{2}(1-2x)^2 + \frac{c_1^2}{2}x^2(1-x)^2 - c_1x(1-x) \right] dx \\ &= \frac{c_1^2}{2} \int_0^1 [1 - 4x + 5x^2 - 2x^3 + x^4] dx + c_1 \int_0^1 [-x + x^2] dx \\ &= \frac{11c_1^2}{60} - \frac{c_1}{6}. \end{aligned}$$

Example 12.1a

- We solve for c_1 by setting

$$\frac{dJ_1}{dc_1} = \frac{11c_1}{30} - \frac{1}{6} = 0$$

to get $c_1 = 5/11$, so the approximate extremal is

$$y_1(x) = \frac{5}{11}x(1-x).$$

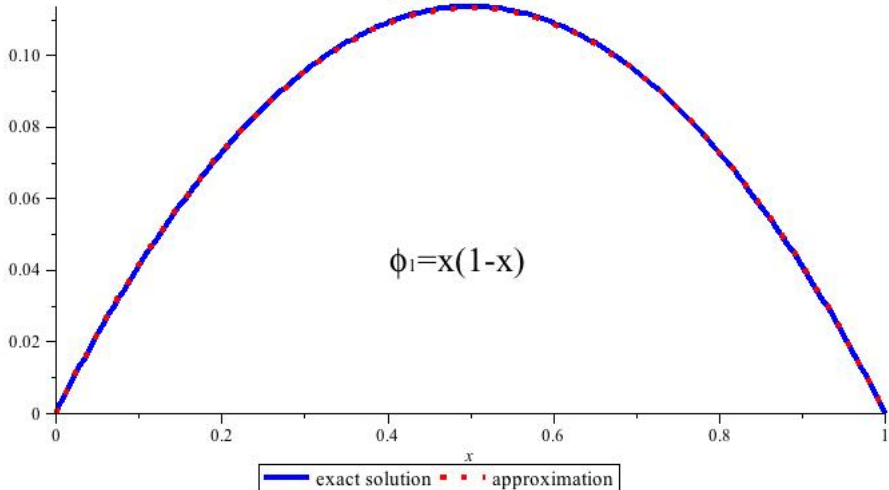
- The value of the approximate functional at this point is

$$J_1[5/11] = \frac{11c_1^2}{60} - \frac{c_1}{6} = -0.37879$$

which is an upper bound on the true value of the functional on the extremal.

Ritz's method

$$\phi_1 = x(1-x)$$



Example 12.1a (alternate approach)

- Choose $\phi_1(x) = \sin(\pi x)$ (use the first element of a trigonometric series to approximate y).
- Then, $\phi_1'(x) = \pi \cos(\pi x)$, and so the functional is

$$\begin{aligned} J_1[c_1] &= J[c_1 \phi_1] = \int_0^1 \left[\frac{1}{2} c_1^2 \phi_1'^2 + \frac{1}{2} c_1^2 \phi_1^2 - c_1 \phi_1 \right] dx \\ &= \int_0^1 \left[\frac{c_1^2 \pi^2}{2} \cos^2(\pi x) + \frac{c_1^2}{2} \sin^2(\pi x) - c_1 \sin(\pi x) \right] dx. \end{aligned}$$

- Observe that $\int_0^1 \cos^2(\pi x) dx = \int_0^1 \sin^2(\pi x) dx = 1/2$, and

$$\int_0^1 \sin(\pi x) dx = \left[-\frac{1}{\pi} \cos(\pi x) \right]_0^1 = -2/\pi.$$

Example 12.1a (alternate approach)

- So

$$J_1[c_1] = \frac{c_1^2}{4} [\pi^2 + 1] - \frac{2}{\pi} c_1.$$

- Once again we solve for c_1 by setting

$$\frac{dJ_1}{dc_1} = \frac{c_1}{2} [\pi^2 + 1] - \frac{2}{\pi} = 0$$

to get $c_1 = \frac{4}{\pi(\pi^2+1)}$, so the approximate extremal is

$$y_1(x) = \frac{4}{\pi(\pi^2 + 1)} \sin(\pi x).$$

