Calculus of Variations
Summer Term 2016

Lecture 15

Universität des Saarlandes

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Purpose of Lesson:

- Hamiltonian’s formulation.
- To introduce Pontryagin’s Maximum Principle (PMP)
- To discuss several PMP examples

Hamiltonian's formulation

- We've seen the Hamiltonian $\mathbb{H}$ earlier an, but haven't explored its full power.
- Using $\mathbb{H}$ can often result in a simpler approach than solving the E-L equatons, e.g., where F has no dependence on x, or where there is more than one dependent variable.
- Hamiltonian's formulation can lead to an understanding of how symmetries in the problem of interest lead to conservation laws.

Legendre transformation

- transformation that depends on the derivatives of a variable
- simple one variable Legendre transform of

$$y : [x_0, x_1] \longrightarrow \mathbb{R},$$

by defining new variable p, by

$$p(x) = y'(x)$$

- provided $y''(x) > 0$ we can define x in terms of p, by introducing the Hamiltonian

$$\mathbb{H}(p) = px - y(x)$$

Legendre transformation

Assume for convenience that y is convex, e.g. $y'' > 0$ for $x \in [x_0, x_1]$. Then

$$\begin{aligned}
\frac{d\mathbb{H}}{dp} &= \frac{d}{dp}(xp) - \frac{dy}{dp} \\
&= p\frac{dx}{dp} + x - \frac{dy}{dp} \\
&= p\frac{dx}{dp} + x - \frac{dy}{dx}\frac{dx}{dp} \\
&= \left(p - \frac{dy}{dx}\right)\frac{dx}{dp} + x \\
&= x
\end{aligned}$$

and also note $px - \mathbb{H} = y$, so from the pair $(p, \mathbb{H})$ we can recover the original pair (x, y), by a Legendre transform.

Hamiltonian's formulation

Refer back to problems with more than one dependent variable, or where F has no dependence on x.

Define generalized coordinates $\mathbf{q} : [t_0, t_1] \to \mathbb{R}^n$.

- i.e. take a set of n functions $q_k(t)$, with two continuous derivatives with respect to t, and put them into a vector $\mathbf{q}(t)$
- dot notation

$$\dot{q}_k = \frac{dq_k}{dt}, \quad \ddot{q}_k = \frac{d^2 q_k}{dt^2} \quad \text{and} \quad \dot{\mathbf{q}} = \left(\frac{dq_1}{dt}, \frac{dq_2}{dt}, \dots, \frac{dq_n}{dt} \right)$$

- Lagrangian $L(t, \mathbf{q}, \dot{\mathbf{q}})$

Hamilton's formulation

The extremal of the functional

$$J[\mathbf{q}] = \int_{t_0}^{t_1} L(t, \mathbf{q}, \dot{\mathbf{q}})dt$$

satisfy the Euler-Lagrange equations

$$\frac{\partial L}{\partial q_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = 0$$

for all k.

Hamilton's formulation

Legendre transform introduces the conjugate variables

$$p_i = \frac{\partial L}{\partial \dot{q}_i}.$$

Suppose these equations can be solved to write $\dot{q}_i$ as a function of (t, q_i, p_i), then the Hamiltonian is

$$\mathbb{H}(t, q_1, \dots, q_n, p_1, \dots, p_n) = \sum_{i=1}^{n} p_i \dot{q}_i - L(t, \mathbf{q}, \dot{\mathbf{q}}).$$

- the p_i are called generalized momenta

Hamilton's formulation

$$\mathbb{H}(t, q_1, \dots, q_n, p_1, \dots, p_n) = \sum_{i=1}^{n} p_i \dot{q}_i - L(t, \mathbf{q}, \dot{\mathbf{q}}).$$

So

$$\frac{\partial \mathbb{H}}{\partial p_i} = \dot{q}_i$$
$$\frac{\partial \mathbb{H}}{\partial q_i} = -\frac{\partial L}{\partial q_i}$$

Given the E-L equations, the second equation gives

$$\frac{\partial \mathbb{H}}{\partial q_i} = -\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = -\frac{dp_i}{dt}.$$

Canonical Euler-Lagrange equations

$$\frac{\partial \mathbb{H}}{\partial p_i} = \frac{dq_i}{dt}$$
$$\frac{\partial \mathbb{H}}{\partial q_i} = -\frac{dp_i}{dt}$$

- called Hamiltonian's equations or canonical Euler-Lagrange equations.
- The n E-L DEs converted into $2n$ first-order DEs
- derivatives are now uncoupled
  - therefore may be easier to solve

Canonical Euler-Lagrange equations

We can get the same canonical E-L equations from finding extremals of the functional of $2n$ variables

$$\widehat{J}[q_1, \dots, q_n, p_1, \dots, p_n] = \int\limits_a^b \left[\sum_{i=1}^n p_i \dot{q}_i - \mathbb{H} \right] dx$$

E.G.

$$\left(\frac{\partial}{\partial q_i} - \frac{d}{dt} \frac{\partial}{\partial \dot{q}_i} \right) \left[\sum_{i=1}^n p_i \dot{q}_i - \mathbb{H} \right] = 0$$

$$\left(\frac{\partial}{\partial p_i} - \frac{d}{dt} \frac{\partial}{\partial \dot{p}_i} \right) \left[\sum_{i=1}^n p_i \dot{q}_i - \mathbb{H} \right] = 0$$

Hamilton's formulation

- J and $\widehat{J}$ are equivalent under the Legendre transformation
  - make q and p independent, whereas before it was a bit of trick to pretend q and $\dot{q}_i$ were independent
- If L does not depend on t, then it should be clear from the Legendre transformation that $\mathbb{H}$ won't depend on t
  - the system will be conservative
  - i.e. $\mathbb{H}$ is a conserved (constant) quantity

Pontryagin’s Maximum Principle

Modern optimal control theory often starts from the PMP. It is a simple, coincise condition for an optimal control.

General control problem

Minimize functional

$$J[\mathbf{x}, \mathbf{u}] = \int_{t_0}^{t_1} F_0(t, \mathbf{x}, \mathbf{u}) dt$$

subject to constraints $\dot{\mathbf{x}} = \mathbf{F}(t, \mathbf{x}, \mathbf{u})$, or more fully,

$$\dot{x}_i = F_i(t, \mathbf{x}, \mathbf{u})$$

- notice no dependence on $\dot{\mathbf{x}}$ in F_0
  - this differs from many CoV problems
- no dependence on $\dot{\mathbf{x}}$ in F_i because we rearrange the equations so that derivatives are on the LHS.

Pontryagin's Maximum Principle (PMP)

Let $\mathbf{u}(t)$ be an admissible control vector that transfers $(t_0, \mathbf{x}_0)$ to a target $(t_1, \mathbf{x}(t_1))$. Let $\mathbf{x}(t)$ be the trajectory corresponding to $\mathbf{u}(t)$.

In order that $\mathbf{u}(t)$ be optimal, it is necessary that there exists $\mathbf{p}(t) = (p_1(t), p_2(t), \dots, p_n(t))$ and a constant scalar p_0 such that

- $\mathbf{p}$ and $\mathbf{x}$ are the solution to the canonical system

$$\dot{\mathbf{x}} = \frac{\partial \mathbb{H}}{\partial \mathbf{p}} \quad \text{and} \quad \dot{\mathbf{p}} = -\frac{\partial \mathbb{H}}{\partial \mathbf{x}}$$

- where the Hamiltonian is $\mathbb{H} = \sum\limits_{i=0}^{n} p_i F_i$ with $p_0 = -1$
- $\mathbb{H}(t, \mathbf{x}, \mathbf{u}, \mathbf{p}) \geqslant \mathbb{H}(t, \mathbf{x}, \widehat{\mathbf{u}}, \mathbf{p})$ for all alternate controls $\widehat{\mathbf{u}}$
- all boundary conditions are satisfied

PMP proof sketch-1

Consider the general problem: minimize functional

$$J[\mathbf{x},\mathbf{u}] = \int_{t_0}^{t_1} F_0(t,\mathbf{x},\mathbf{u})dt$$

subject to constraints

$$\dot{x}_i = F_i(t,\mathbf{x},\mathbf{u}).$$

We can incorporate the constraints into the functional using the Lagrange multipliers λ_i, e.g.

$$\begin{aligned}\widehat{J} &= \int_{t_0}^{t_1} L(t,\mathbf{x},\dot{\mathbf{x}},\mathbf{u})dt \\ &= \int_{t_0}^{t_1} F_0(t,\mathbf{x},\mathbf{u})dt + \sum_{i=1}^{n} \lambda_i(t)\left[\dot{x}_i - F_i(t,\mathbf{x},\mathbf{u})\right] dt\end{aligned}$$

PMP proof sketch-2

Given such a function we get (by definition)

$$p_i = \frac{\partial L}{\partial \dot{x}_i} = \lambda_i .$$

So we can identify the Lagrange multipliers λ_i with the generalized momentum terms p_i

1. the p_i are known in economics literature as marginal valuation of x_i or the shadow prices
2. shows how much a unit increment in x at time t contributes to the optimal objective functional $\widehat{J}$
3. the p_i are known in control as co-state variables (sometimes written as z_i).

PMP proof sketch-3

By definition the Hamiltonian is

$$\begin{aligned}\mathbb{H}(t,\mathbf{x},\mathbf{p},\mathbf{u}) &= \sum_{i=1}^{n} p_i\dot{x}_i - L(t,\mathbf{x},\dot{\mathbf{x}},\mathbf{p},\mathbf{u})\\ &= \sum_{i=1}^{n} p_i\dot{x}_i - F_0(t,\mathbf{x},\mathbf{u}) - \sum_{i=1}^{n}\lambda_i(t)\left[\dot{x}_i - F_i(t,\mathbf{x},\mathbf{u})\right]\\ &= -F_0(t,\mathbf{x},\mathbf{u}) + \sum_{i=1}^{n} p_i F_i(t,\mathbf{x},\mathbf{u})\end{aligned}$$

because $\lambda_i = p_i$, so the $\dot{x}_i$ terms cancel. The final result is just the Hamiltonian as defined in the PMP.

PMP proof sketch-4

From previous slide the Hamiltonian can be written

$$\mathbb{H}(t, \mathbf{x}, \mathbf{p}, \mathbf{u}) = -F_0(t, \mathbf{x}, \mathbf{u}) + \sum_{i=1}^{n} p_i F_i(t, \mathbf{x}, \mathbf{u})$$

which is the Hamiltonian defined in the PMP. Then the canonical E-L equations (Hamilton's equations) are

$$\frac{\partial \mathbb{H}}{\partial p_i} = \frac{dx_i}{dt} \quad \text{and} \quad \frac{\partial \mathbb{H}}{\partial x_i} = -\frac{dp_i}{dt}.$$

Note that the equations $\frac{\partial \mathbb{H}}{\partial p_i} = \frac{dx_i}{dt}$ just revert to

$$F_i(t, \mathbf{x}, \mathbf{u}) = \dot{x}_i$$

which are just the system equations.

PMP proof sketch-5

Finally, note that Hamilton's equations above only relate x_i and p_i. What about equations for u_i?

Take the conjugate variable to be z_i, and we get (by definition) that

$$z_i = \frac{\partial L}{\partial \dot{u}_i} = 0$$

and the second of Hamilton's equations is therefore

$$\frac{\partial \mathbb{H}}{\partial u_i} = -\frac{dz_i}{dt} = 0$$

which suggests a stationary point of $\mathbb{H}$ WRT u_i.

In fact we look for a maximum (and note this may happen on the bounds of u_i).

PMP Example: plant growth

Example 15.1 (Plant growth-1)

Plant growth problem:

- market gardener wants to plants to grow to a fixed height 2 within a fixed window of time $[0,1]$
- can supplement natural growth with lights (at night)
- growth rate dictates

$$\dot{x} = 1 + u$$

- cost of lights

$$J[u] = \int_0^1 \frac{1}{2} u^2 dt$$

PMP Example: plant growth

Example 15.1 (Plant growth-2)

Minimize

$$J[u] = \int_0^1 \frac{1}{2} u^2 dt$$

subject to $x(0) = 0$ and $x(1) = 2$ and

$$\dot{x} = F_1(t, x, u) = 1 + u.$$

Hamiltonian is

$$\begin{aligned} \mathbb{H} &= -F_0(t, x, u) + pF_1(t, x, u) \\ &= -\frac{1}{2}u^2 + p\,(1 + u)\,. \end{aligned}$$

PMP Example: plant growth

Example 15.1 (Plant growth-3)

Hamiltonian is

$$\mathbb{H} = -\frac{1}{2}u^2 + p\,(1+u)\,.$$

Canonical equations

$$\frac{\partial \mathbb{H}}{\partial p} = \frac{dx}{dt} \quad \text{and} \quad \frac{\partial \mathbb{H}}{\partial x} = -\frac{dp}{dt}$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$1 + u = \dot{x} \qquad\qquad 0 = -\dot{p}$$

LHS $\Rightarrow$ system DE

RHS $\Rightarrow$ $\dot{p} = 0$ means that $p = c_1$ where c_1 is a constant.

PMP Example: plant growth

Example 15.1 (Plant growth-4)

Maximum principle requires $\mathbb{H}$ be a maximum, for which

$$\frac{\partial \mathbb{H}}{\partial u} = -u + p = 0.$$

So $u = p$, and $\dot{x} = 1 + u$ so

$$x = (1 + c_1)\, t + c_2.$$

The solution which satisfies $x(0) = 0$ and $x(1) = 2$ is

$$x = 2t.$$

So $u = c_1 = 1$, and the optimal cost is $\frac{1}{2}$.

PMP and natural boundary conditions

Typically we fix t_0 and $\mathbf{x}(t_0)$, but often the right-hand boundary condition is not fixed, so we need natural boundary conditions.

Here, they differ from traditional CoV problems in two respects:

- The terminal cost ϕ
- The function F_0 is not explicitly dependent on $\dot{x}$.

The resulting natural boundary conditions are

$$\sum_i \left(\frac{\partial \phi}{\partial x_i} + p_i \right) \delta x_i \Big|_{t=t_1} + \left(\frac{\partial \phi}{\partial t} - \mathbb{H} \right) \delta t \Big|_{t=t_1} = 0$$

for all allowed δx_i and δt.

PMP and natural boundary conditions

The resulting natural boundary condition is

$$\sum_i \left(\frac{\partial \phi}{\partial x_i} + p_i \right) \delta x_i \Big|_{t=t_1} + \left(\frac{\partial \phi}{\partial t} - \mathbb{H} \right) \delta t \Big|_{t=t_1} = 0.$$

Special cases

- when t_1 is fixed and $\mathbf{x}(t_1)$ is completely free we get

$$\left(\frac{\partial \phi}{\partial x_i} + p_i \right) \delta x_i \Big|_{t=t_1} = 0, \quad \forall i$$

- when $\mathbf{x}(t_1)$ is fixed, $\delta x_i = 0$, and we get

$$\left(\frac{\partial \phi}{\partial t} - \mathbb{H} \right) \delta t \Big|_{t=t_1} = 0.$$

Example: stimulated plant growth

Example 15.2 (Stimulated plant growth-1)

Plant growth problem:

- market gardener wants to plants to grow as much as possible within a fixed window of time $[0, 1]$
- supplement natural growth with lights as before
- growth rate dictates $\dot{x} = 1 + u$
- cost of lights

$$J[u] = \int_0^1 \frac{1}{2} u^2(t) dt$$

- value of crop is proportional to the height

$$\phi(t_1, \mathbf{x}(t_1)) = x(t_1).$$

Plant growth problem statement

Example 15.2 (Stimulated plant growth-2)

Write as a minimization problem

$$J[x,u] = -x(t_1) + \int_0^1 \frac{1}{2}u^2 dt$$

subject to $x(0) = 0$, and

$$\dot{x} = 1 + u.$$

- the terminal cost doesn't affect the shape of the solution
- but we need a natural end-point condition for t_1.

Plant growth: natural BC

Example 15.2 (Stimulated plant growth-3)

The problem is solved as before, but we write the natural boundary condition at $x = t_1$ as

$$\left(\frac{\partial \phi}{\partial x_i} + p_i \right) \Big|_{t=t_1} = 0, \qquad \forall i$$

which reduces to

$$-1 + p\big|_{t=t_1} = 0.$$

Given p is constant, this sets $p(t) = 1$, and hence the control $u = 1$ (as before).

Autonomous problems

Autonomous problems have no explicit dependence on t.

- time invariance symmetry
- hence $\mathbb{H}$ is constant along the optimal trajectory
- if the end-time is free (and the terminal cost is zero) then the transversality conditions ensure $\mathbb{H} = 0$ along the optimal trajectory.