Calculus of Variations
Summer Term 2016

Lecture 17

Universität des Saarlandes

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Purpose of Lesson:

- To introduce direct methods
- To discuss strong and weak topologies in real Banach spaces.

Direct Methods:

- The relevant ideas developed during the 20th century are called Direct Methods.
- An important ingredient here is the introduction of functional analytic techniques.
- In fact, it was the Calsulus of Variations, which gave birth to the theory of Functional Analysis.

The Main Idea of Direct Method:

Consider a minimization problem on some class $\mathcal{A}$ of functions:

$$\min_{u\in\mathcal{A}} J[u]$$

- Suppose there exist a minimizing sequence $\{u_n\}$ in $\mathcal{A}$, i.e.

$$\lim_{n\to\infty} J[u_n] = \min_{u\in\mathcal{A}} J[u] < +\infty.$$

- Suppose we could find an element u_0 in $\mathcal{A}$ such that

$$''\ u_n \to u_0\ '' \quad \text{as} \quad n \to \infty$$

- Suppose the functional J has some kind of continuity. Therefore

$$\lim_{n\to\infty} J[u_n] = J[u_0].$$

- We conclude that u_0 is a minimizer of the functional J because

$$J[u_0] = \lim_{n\to\infty} J[u_n] = \min_{u\in\mathcal{A}} J[u].$$

Example 17.1 (J is not bounded below)

Let $[0,\pi] \subset \mathbb{R}$ and $\mathcal{A} = \left\{ u \in C^1\left([0,\pi]\right) : u(0) = u(\pi) = 0 \right\}$.

Take

$$J[u] = \int\limits_0^\pi \left[u'^2 - 2u^2 \right] dx \qquad \text{and} \qquad u_n(x) = n \sin x \in \mathcal{A}.$$

But $J[u_n] = -\dfrac{\pi n^2}{2} \to -\infty$ as $n \to \infty$. (Also, u_n is unbounded.)

Example 17.2 (No convergent subsequence in minimizing sequence)

Let $[0,\pi] \subset \mathbb{R}$ and $\mathcal{A} = \left\{ u \in C^1([0,\pi]) : u(0) = u(\pi) = 0 \right\}$.

Take

$$J[u] = \int\limits_0^\pi \left(u'^2 - 1 \right)^2 dx,$$

$$u_n(x) = \sqrt{\frac{1}{n^2} + \frac{\pi^2}{4}} - \sqrt{\frac{1}{n^2} + \left(x - \frac{\pi}{2} \right)^2} \in \mathcal{A},$$

$$J[u_n] \to 0 = \inf_{v \in \mathcal{A}} J[v].$$

But every subsequence converges to $\frac{\pi}{2} - |x - \frac{\pi}{2}|$ which is not in $\mathcal{A}$.

Example 17.3 (J is not lower semi-continuous)

Let $[0,\pi] \subset \mathbb{R}$ and $\mathcal{A} = \left\{u \in C^1([0,\pi]) : u(0) = u(\pi) = 0\right\}$.

Let

$$g(p) = \begin{cases} p^2, \text{ if } p \neq 0; \\ 1, \text{ if } p = 0. \end{cases}$$

Take

$$J[u] = \int_0^\pi g(u')dx \quad \text{and} \quad u_n(x) = \frac{1}{n}\sin x \to 0.$$

Then $u_n(x) \in \mathcal{A}$ and $J[u_n] = \dfrac{\pi}{2n^2} \to 0 = I$ as $n \to \infty$ but

$$\pi = J[0] = J[\lim_{n' \to \infty} u_{n'}] > \lim_{n \to \infty} J[u_n] = 0.$$

§13. Suitable Function Spaces

Topologies on Banach Spaces

- Let $(\mathbb{X}, |\cdot|)$ denote a real Banach space.
- A Banach space is a complete, normed linear space. Complete means that any Cauchy sequence is convergent.
- A sequence $\{x_n\}$ of real numbers is called a Cauchy sequence, if for every positive real number ε, there is a positive integer N such that for all natural numbers $m, n > N$

$$|x_n - x_m| < \varepsilon.$$

- Examples:
  1. $\mathbb{R}^n$ with the norm defined by $\|x\| = \left(\sum\limits_i |x_i|^2\right)^{1/2}$.
  2. $C^k\left(\overline{\Omega}\right)$ with the norm defined by

$$\|u\|_{C^k} = \max_{0\leqslant|\alpha|\leqslant k} \sup_{x\in\overline{\Omega}} |D^\alpha u(x)|$$

- We denote by $\mathbb{X}'$ the topological dual space of $\mathbb{X}$:

$$\mathbb{X}' = \left\{ l : \mathbb{X} \to \mathbb{R} \text{ linear such that } |l|_{\mathbb{X}'} = \sup_{x \neq 0} \frac{|l(x)|}{|x|_{\mathbb{X}}} < \infty \right\}.$$

- Classically, $\mathbb{X}$ can be endowed with two topologies.

Definition (topologies on $\mathbb{X}$)

(i) The strong topology, denoted by $x_n \underset{\mathbb{X}}{\to} x$, is defined by

$$|x_n - x|_{\mathbb{X}} \to 0 \qquad (n \to \infty).$$

(ii) The weak topology, denoted by $x_n \underset{\mathbb{X}}{\rightharpoonup} x$, is defined by

$$l(x_n) \to l(x) \qquad (n \to \infty) \quad \text{for every} \quad l \in \mathbb{X}'.$$

- Strong convergence implies weak convergence, but the converse is false in general.

Example 17.4 (Counterexample)

- Consider the sequence $f_n(x) = \sin(2\pi x n)$, $x \in (0, 1)$, as $n \to \infty$.
- $f_n \underset{L_2(\Omega)}{\rightharpoondown} 0$. To prove weak convergence, we have for all $\varphi \in C^1(0, 1)$, thanks to a classical integration by parts,

$$\int_0^1 \sin(2\pi x n)\varphi(x)dx = \frac{1}{2\pi n}[\varphi(0) - \varphi(1)] + \frac{1}{2\pi n}\int_0^1 \cos(2\pi x n)\varphi'(x)dx$$

Example 17.4 (continued)

- So, it is clear that $\langle f_n, \varphi \rangle \to 0$ as $n \to \infty$ for all $\varphi \in C^1(0,1)$, where $\langle \cdot, \cdot \rangle$ is the usual scalar product in $L_2(0,1)$.
- By density, this result can be generalized for all $\varphi \in L_2(0,1)$.
- There is no strong convergence, i.e., $\boxed{f_n(x) \underset{L_2(0,1)}{\longrightarrow} 0}$ is not true.
- To prove that there is no strong convergence, we observe that

$$\int\limits_0^1 \sin^2(2\pi x n) dx = \frac{1}{2} \int\limits_0^1 (1 - \cos[(4\pi x n)) \, dx = \frac{1}{2}.$$

- The dual space $\mathbb{X}'$ can also be endowed with the strong and the weak topologies.

Definition (topologies on $\mathbb{X}'$)

(i) The strong topology, denoted by $l_n \underset{\mathbb{X}'}{\rightarrow} l$, is defined by

$$|l_n - l|_{\mathbb{X}'} \rightarrow 0, \text{ or equivalently, } \sup_{x \neq 0} \frac{|l_n(x) - l(x)|}{|x|_{\mathbb{X}}} \rightarrow 0 \quad (n \rightarrow \infty).$$

(ii) The weak topology, denoted by $l_n \underset{\mathbb{X}'}{\rightharpoonup} l$, is defined by

$$z(l_n) \rightarrow z(l) \quad (n \rightarrow \infty) \quad \text{for every} \quad z \in \left(\mathbb{X}'\right)',$$

where $(\mathbb{X}')'$ denotes the bidual space of $\mathbb{X}$.

- In some cases it is more convenient to equip $\mathbb{X}'$ with a third topology:

  The weak* topology, denoted by $l_n \underset{\mathbb{X}'*}{\rightharpoonup} l$, is defined by

$$l_n(x) \to l(x) \quad (n \to \infty) \quad \text{for every} \quad x \in \mathbb{X}.$$

- The space $\mathbb{X}$ is called reflexive if $(\mathbb{X}')' = \mathbb{X}$.
- $\mathbb{X}$ is called separable if it contains a countable dense set.
- Examples:
  1. $\mathbb{X} = L_p(\Omega)$ is reflexive for $1 < p < \infty$ and separable for $1 \leqslant p < \infty$. The dual space of $L_p(\Omega)$ is $L_{p'}(\Omega$ for $1 \leqslant p < \infty$ with $\dfrac{1}{p} + \dfrac{1}{p'} = 1$.
  2. $\mathbb{X} = L_1(\Omega)$ is nonreflexive and $\mathbb{X}' = L_\infty(\Omega)$.

The main properties associated with these different topologies are summarized in the following theorem :

Theorem 17.1 (weak sequential compactness)

(i) Let $\mathbb{X}$ be a reflexive Banach space, $K > 0$, and $x_n \in \mathbb{X}$ a sequence such that $|x_n|_{\mathbb{X}} \leqslant K$.

Then there exist $x \in \mathbb{X}$ and a subsequence x_{n_j} of x_n such that

$$x_{n_j} \underset{\mathbb{X}}{\rightharpoonup} x \qquad (n \to \infty).$$

(ii) Let $\mathbb{X}$ be a separable Banach space, $K > 0$, and $l_n \in \mathbb{X}'$ such that $|l_n|_{\mathbb{X}'} \leqslant K$.

Then there exist $l \in \mathbb{X}'$ and a subsequence l_{n_j} of l_n such that

$$l_{n_j} \underset{\mathbb{X}'*}{\rightharpoonup} l \qquad (n \to \infty).$$