Calculus of Variations
Summer Term 2016

Lecture 18

Universität des Saarlandes

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Purpose of Lesson:

- To introduce Sobolev spaces which would be suitable function spaces in direct methods
- To introduce the notions of convexity and lower semicontinuity and find the connection between these notions.
- To formulate the existence theorem.
- To discuss several examples emphasizing the optimality of the hypothesis of the existence theorem.

Sobolev spaces

- Before giving the definition of Sobolev spaces, we need to weaken the notion of derivative.
- In doing so we want to keep the right to intergate by parts; this is one of the reasons of the following definition.

Definition (weak derivative)

Let $\Omega \subset \mathbb{R}^n$ be open and $u \in L_{1,loc}(\Omega)$.

We say that $v \in L_{1,loc}(\Omega)$ is the weak partial derivative of u with respect to x_i if

$$\int_{\Omega} v(x)\varphi(x)dx = -\int_{\Omega} u(x)\frac{\partial \varphi(x)}{\partial x_i}dx, \quad \forall \varphi \in C_0^{\infty}(\Omega).$$

By abuse of notations we write $v = \dfrac{\partial u}{\partial x_i}$ or u_{x_i}.

We say that u is weakly differentiable of the weak partial derivatives $u_{x_1}, \dots, u_{x_n}$ exist.

Remarks

- All the usual rules of differentiation are easily generalized to the present context of weak differentiability.
- If a function is C^1, then the usual notion of derivative and the weak one coincide.
- Not all measurable functions can be differentiated weakly. In particular, a discontinuous function of $\mathbb{R}$ cannot be differentiated in the weak sense.

Definition (Sobolev spaces)

Let $\Omega \subset \mathbb{R}^n$ be an open set and $1 \leqslant p \leqslant \infty$

We let $W_p^1(\Omega)$ be the set of functions $u : \Omega \to \mathbb{R}$, $u \in L_p(\Omega)$, whose weak partial derivatives $u_{x_i} \in L_p(\Omega)$ for every $i = 1, \dots, n$.

We endow this space with the following norm

$$\|u\|_{W_p^1} = \left(\|u\|_p^p + \|\nabla u\|_p^p\right)^{1/p} \quad \text{if} \quad 1 \leqslant p < \infty$$
$$\|u\|_{W_\infty^1} = \max\left\{\|u\|_\infty, \|\nabla u\|_\infty\right\} \quad \text{if} \quad p = \infty.$$

Here

$$\|u\|_p := \left(\int\limits_\Omega |u|^p dx\right)^{1/p} \quad \text{if} \quad 1 \leqslant p < \infty$$

$$\|u\|_\infty := \operatorname*{ess\,sup}_{x \in \Omega} |u(x)| = \inf\left\{\alpha : |u(x)| \leqslant \alpha \text{ a.e.in } \Omega\right\}.$$

Remarks

- By abuse of notations we write $W_p^0 = L_p$.
- If $1 \leqslant p < \infty$, the set $W_{p,0}^1(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$-functions in $W_p^1(\Omega)$.
- We often say, if Ω is bounded, that $u \in W_{p,0}^1(\Omega)$ is such that $u \in W_p^1(\Omega)$ and $u = 0$ on $\partial\Omega$.
- We also write $u \in u_0 + W_{p,0}^1(\Omega)$ meaning that $u, u_0 \in W_p^1(\Omega)$ and $u - u_0 \in W_{p,0}^1(\Omega)$.
- We let $W_{\infty,0}^1(\Omega) = W_\infty^1(\Omega) \cap W_{1,0}^1(\Omega)$.
- Note that if Ω is bounded, then

$$C^1(\overline{\Omega}) \subsetneqq W_\infty^1(\Omega) \subsetneqq W_p^1(\Omega) \subsetneqq L_p(\Omega) \qquad \text{for every} \quad 1 \leqslant p < \infty.$$

Analogously we define the Sobolev spaces with higher derivatives as folows.

Definition (Sobolev spaces with higher derivatives)

Let $\Omega \subset \mathbb{R}^n$ be an open set and $1 \leqslant p \leqslant \infty$

If $k > 0$ is an integer we let $W_p^k(\Omega)$ to be the set of functions $u : \Omega \to \mathbb{R}$, whose weak partial derivatives $D^\alpha u \in L_p(\Omega)$, for every multi-index $\alpha \in \mathcal{A}_m$ with

$$\mathcal{A}_m := \left\{ \alpha = (\alpha_1, ...\alpha_n), \alpha_j \geqslant 0 \text{ an integer and } \sum_{j=1}^{n} \alpha_j = m \right\},$$

$0 \leqslant m \leqslant k$.

Definition (Sobolev spaces with higher derivatives - cont.)

The norm in $W_p^k(\Omega)$ is given by

$$\|u\|_{W_p^k} = \begin{cases} \left(\sum\limits_{0 \leqslant |\alpha| \leqslant k} \|D^\alpha u\|_p^p \right)^{1/p} & \text{if} \quad 1 \leqslant p < \infty \\ \max\limits_{0 \leqslant |\alpha| \leqslant k} \{ \|D^\alpha u\|_\infty \} & \text{if} \quad p = \infty \end{cases}$$

Remark

If we denote by $I = (a, b)$, we have, for $p \geqslant 1$,

$$C_0^\infty(I) \subset \dots \subset W_p^2(I) \subset C^1(\bar{I}) \subset W_p^1(I) \subset C(\bar{I}) \subset L_\infty(I) \subset \dots \subset L_2(I) \subset L_1(I)$$

Theorem 18.1

Let $\Omega \subset \mathbb{R}^n$ be an open set, $1 \leqslant p \leqslant \infty$ and $k \geqslant 1$ an integer.

$W_p^k(\Omega)$ equipped with its norm $\| \cdot \|_{W_p^k}$ is a Banach space which is separable if $1 \leqslant p < \infty$ and reflexive if $1 < p < \infty$.

Remarks

- Note that the space W_1^1 is not reflexive.
- This is the main source of difficulties in the minimal surface problem.

§14. Convexity and Lower Semicontinuity

Let $\mathbb{X}$ be a Banach space, $J : \mathbb{X} \to \mathbb{R}$, and consider the minimization problem

$$\inf_{u \in \mathbb{X}} J[u].$$

- Let us first consider the problem of the existence of a solution.
- Proving of existence is usually achieved by the following steps, which constitute the direct method of the calculus of variations:

1. One constructs a minimizing sequence $u_n \in \mathbb{X}$, i.e., a sequence satisfying

$$\lim_{n\to\infty} J[u_n] = \inf_{u\in\mathbb{X}} J[u].$$

2. If J is coercive $\left(\lim_{|u|\to\infty} J[u] = \infty\right)$, one can obtain a uniform bound $|u_n|_{\mathbb{X}} \leqslant C$. If $\mathbb{X}$ is reflexive, then (by Theorem 18.1) one deduce the existence of $u_0 \in \mathbb{X}$ and of subsequence u_{n_j} such that

$$u_{n_j} \underset{\mathbb{X}}{\rightharpoonup} u_0.$$

3. To prove that u_0 is a minimum point of J it suffices to have the inequality

$$\underline{\lim}_{u_{n_j}\rightharpoonup u_0} J[u_{n_j}] = \sup \inf_{u_{n_j}\rightharpoonup u_0} J[u_{n_j}] \geqslant J[u_0],$$

which obviously implies that $J[u_0] = \min_{u\in\mathbb{X}} J[u]$.

Lower Semicontinuity

This last property, which appears here naturally, is called weak lower semicontinuity. More precisely, we have the following definition:

Definition (lower semicontinuity)

- J is called lower semicontinuous (l.s.c.) for the weak topology if for all sequence $u_n \rightharpoonup u_0$ we have

$$\varliminf_{u_n \rightharpoonup u_0} J[u_n] = \sup \inf_{u_n \rightharpoonup u_0} J[u_n] \geqslant J[u_0]$$

- The same definition can be given with a strong topology.

- In the direct method, the notion of weak l.s.c. emerges very naturally.

Convexity

Unfortunately, it is difficult in general to prove weak l.s.c.

A sufficient condition that implies weak l.s.c. is convexity:

Definition (convexity)

J is convex on $\mathbb{X}$ if

$$J[\lambda u + (1-\lambda)v] \leqslant \lambda J[u] + (1-\lambda)J[v]$$

for all $u, v \in \mathbb{X}$ and $\lambda \in [0,1]$.

Convexity and Integral Functionals

If J is an integral functional, we can even say more about the link between convexity and l.s.c.

- Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, and let $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a continuous function satisfying
$$0 \leqslant f(x, u, \xi) \leqslant a(x, |u|, |\xi|),$$
where a is increasing with respect to $|u|$ and $|\xi|$, and integrable in x.
- Let $W^1_p(\Omega)$ be the Sobolev space.
- For $u \in W^1_p(\Omega)$ we consider the functional
$$J[u] = \int\limits_{\Omega} f(x, u, \nabla u) dx$$

Convexity and Integral Functionals

Theorem 18.2 (l.s.c. and convexity)

1.
$$\boxed{\begin{array}{c} J[u] \text{ is (sequentially) weakly l.s.c. on } W_p^1(\Omega), \quad 1 \leqslant p < \infty \\ \Updownarrow \\ f \text{ is convex in } \xi. \end{array}}$$

2.
$$\boxed{\begin{array}{c} J[u] \text{ is (sequentially) weakly* l.s.c. on } W_\infty^1(\Omega), \\ \Updownarrow \\ f \text{ is convex in } \xi. \end{array}}$$

We emphasize that convexity is a sufficient condition for existence. There exist nonconvex problems admitting a solution.

Theorem 18.3

Let $\Omega \subset \mathbb{R}^n$ be bounded and $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ continuous satisfying

1. $f(x,u,\xi) \geqslant a(x) + b|u|^q + c|\xi|^p$ for every (x,u,ξ) and for some $a \in L_1(\Omega)$, $b > 0$, $c > 0$, and $p > q \geqslant 1$.
2. $\xi \to f(x,u,\xi)$ is convex every (x,u).
3. There exists $u_0 \in W^1_p(\Omega)$ such that $J[u_0] < \infty$.

Then the problem

$$\inf \left\{ J[u] = \int\limits_{\Omega} f(x,u(x),\nabla u(x))dx, \quad u \in W^1_p(\Omega) \right\}$$

admits a solution. Moreover, if $(u,\xi) \to f(x,u,\xi)$ is strictly convex for every x, then the solution is unique.

Remarks

- In Theorem 18.3 the coercivity condition (1) implies the boundedness of the minimizing sequences.
- Condition (2) permits us to pass to the limit on these sequences.
- Condition (3) ensures that the problem has a meaning.

We propose below some classical examples where either coercivity, reflexivity, or convexity is no longer true.

Example 18.1 (Weierstrass)

- Let $\Omega = (0, 1]$, let f be defined by $f(x, u, \xi) = x\xi^2$ and let us set

$$J[u] = \int_0^1 x u'^2 dx \qquad \text{with} \quad u(0) = 1, \quad u(1) = 0.$$

- Then, we can show that this problem does not have any solution.

Example 18.1 (continued)

- The function f is convex, but the $W_2^1(\Omega)$-coercivity with respect to u is not satisfied because the integrand $f(x,\xi) = x\xi^2$ vanishes at $x = 0$.

- Let us first prove that

$$m := \inf J[u] = 0.$$

The idea is to propose the following minimizing seqience

$$u_n(x) = \begin{cases} 1, & \text{if} \quad x \in (0, 1/n), \\ -\dfrac{\log(x)}{\log(n)}, & \text{if} \quad x \in (1/n, 1). \end{cases}$$

Example 18.1 (continued)

- It is then easy to verify that $u_n \in W^1_\infty(0,1)$, and that

$$J[u_n] = \int_0^1 x u_n'^2 dx = \frac{1}{\log(n)} \to 0.$$

So we have $m = 0$.

- If there exists a minimum u_0, then we should have $J[u_0] = 0$, that is $u_0' = 0$ almost everywhere (a.e.) in $(0,1)$.
- But $u_0' = 0$ a.e. in $(0,1)$ is clearly incompatible with the boundary conditions.

Example 18.2 (Bolza)

- Let $\Omega = (0, 1]$, and let f be defined by $f(x, u, \xi) = u^2 + (\xi^2 - 1)^2$. The Bolza problem is

$$\min \leftarrow J[u] = \int_0^1 \left[(1 - u'^2)^2 + u^2\right] dx, \quad u \in W_4^1(0, 1)$$

$$\text{with} \quad u(0) = u(1) = 0.$$

- The functional J is clearly nonconvex.

Example 18.2 (continued)

- It is easy to see that

$$m := \inf J[u] = 0.$$

Indeed, for n an integer and $0 \leqslant k \leqslant n-1$, if we choose

$$u_n(x) = \begin{cases} x - \dfrac{k}{n}, & \text{if} \quad x \in \left(\dfrac{2k}{2n}, \dfrac{2k+1}{2n}\right) \\ -x + \dfrac{k+1}{n}, & \text{if} \quad x \in \left(\dfrac{2k+1}{2n}, \dfrac{2k+2}{2n}\right) \end{cases},$$

then $u_n \in W^1_\infty(0,1)$ and

$$0 \leqslant u_n(x) \leqslant \frac{1}{2n} \quad \text{for every } x \in (0,1),$$
$$|u_n'(x)| = 1 \quad \text{a.e. in } (0,1),$$
$$u_n(0) = u_n(1) = 0.$$

Example 18.2 (continued)

- Therefore,

$$0 \leqslant \inf_u J[u] \leqslant J[u_n] \leqslant \frac{1}{4n^2}.$$

  Letting $n \to \infty$, we obtain $m = 0$.

- However, there exists no function $u \in W_4^1(0,1)$ for which $u(0) = u(1) = 0$ and $J[u] = 0$.

- So the problem does not have a solution in $\overset{\circ}{W}{}_{4,0}^1(0,1)$.