Calculus of Variations
Summer Term 2016

Lecture 3

Universität des Saarlandes

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Purpose of Lesson:

- To continue the discussion about special cases of the E-L equation.
- To show that the E-L equation is a necessary, but not sufficient condition for a local extremum.
- To discuss the simplest variational problems involving undetermined end points.

Special Cases of the E-L Equation (cont.)

3. If F does not depend on y', the E-L equation takes the form

$$F_y(x, y) = 0,$$

and hence is not a differential equation, but a *finite*, whose solution consists of one or more curves $y = y(x)$

4. In a variety of problems, one encounters functionals of the form

$$J[y] = \int_a^b f(x,y)\sqrt{1+y'^2}\,dx,$$

representing the integral of a function $f(x,y)$ with respect to the arc length s ($ds = \sqrt{1+y'^2}dx$).

- In this case, the E-L equation can be transformed into

$$F_y - \frac{d}{dx}F_{y'} = f_y(x,y)\sqrt{1+y'^2} - \frac{d}{dx}\left[f(x,y)\frac{y'}{\sqrt{1+y'^2}}\right]$$
$$= \frac{1}{\sqrt{1+y'^2}}\left[f_y - f_x y' - f\frac{y''}{1+y'^2}\right] = 0$$

i.e.,

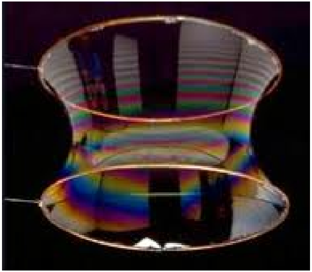
$$\boxed{f_y - f_x y' - f\frac{y''}{1+y'^2} = 0}.$$

Failure of Sufficiency

Remark

Note that the Euler-Lagrange Equation is a necessary, but not sufficient condition for a local extremum.

Next we will consider the famous example of the minimal surface area for a soap film.



Example 3.1

We want to minimize the following functional

$$J[y] = 2\pi \int_{x_0}^{x_1} y\sqrt{1+(y')^2}dx \to \min$$

according to the boundary conditions $y(x_0) = y_0$, $y(x_1) = y_1$.

- If we use the Euler-Lagrange equation and solve it for $y(x)$ we find the Catenary function

$$y(x) = C_1 \cosh\left\{\left(\frac{x+C_2}{C_1}\right)\right\}$$

- Consider the special case where $C_2 = 0$ and require that $y(x) = C_1 \cosh\left\{\left(\frac{x}{C_1}\right)\right\}$ pass through $(-x_1, 1)$ and $(x_1, 1)$ where x_1 is a constant.
- So C_1 will satisfy

$$1 = C_1 \cosh\left\{\left(\frac{x}{C_1}\right)\right\} \tag{3.1}$$

- Compare $y = 1$ and (3.1) versus C_1
  1. For $x_1 = 1$ there is **No Solutions**;
  2. For $x_1 = 0.7$ there is exactly One Solutions;
  3. For $x_1 = 0.4$ there are Two Solutions.

§3. Undetermined End Points

Undetermined End Points

Problem 3-1

We seek to minimize the integral

$$J[y] = \int_a^b F(x, y, y')dx \to \text{ min}$$

with respect to functions that attain the value A for $x = a$, but for which no value is prescribed at $x = b$.

Question:

What is the arc of quickest descent from a fixed point to a vertical line?

- To find the minimizing function, we as before introduce a small variation from $y(x)$, namely
$$y(x) + \varepsilon\eta(x)$$
where ε is a small parameter and $\eta(x)$ is a smooth curve satisfying the BC
$$\eta(a) = 0.$$
- We take the derivative of
$$\phi(\varepsilon) = J[y + \varepsilon\eta]$$
with respect to ε, evaluate it $\varepsilon = 0$, and set this equal to zero; that is,
$$\frac{d\phi(\varepsilon)}{d\varepsilon} = \frac{d}{d\varepsilon} J[y + \varepsilon\eta]\big|_{\varepsilon=0} = \int_a^b \left[\frac{\partial F}{\partial y}\eta(x) + \frac{\partial F}{\partial y'}\eta'(x)\right] dx$$

- Integration by parts gives

$$\left[\frac{\partial F}{\partial y'}\eta(x)\right]^{x=b} + \int_a^b \left\{\frac{\partial F}{\partial y} - \frac{d}{dx}\left[\frac{\partial F}{\partial y'}\right]\right\}\eta(x)dx = 0. \qquad (3.2)$$

- Since (3.2) must hold for all choices of $\eta(x)$ satisfying

$$\eta(a) = 0,$$

it must in particular hold for those η for which $\eta(b) = 0$. For such $\eta(x)$ the first term in (3.2) disappeared and as before we end up with

$$\boxed{\frac{\partial F}{\partial y} - \frac{d}{dx}\left[\frac{\partial F}{\partial y'}\right] = 0}. \qquad (3.3)$$

- So, with result (3.3), and for general $\eta(x)$ once again, the second member of (3.2) reduces to its first term

$$\left[\frac{\partial F}{\partial y'}\eta(x)\right]^{x=b} = 0.$$

- Now, by choosing $\eta(b) = 1$, the vanishing for all η of the term remaining requires fulfillment of the end-point condition

$$\boxed{\left.\frac{\partial F}{\partial y'}\right|_{x=b} = 0}. \qquad (3.4)$$

- The two constants of integration obtained in the solution of (3.3), a second-order equation, are determined by the end-point condition $y(a) = A$ and (3.4) - provided, of course, a solution of the problem exists.

Problem 3-2

We seek to minimize the integral

$$J[y] = \int_a^{x^*} F(x, y, y')dx \to \min$$

with respect to functions which attain the value A for $x = a$ and which satisfy the given relation

$$g(x, y) = 0$$

at the upper limit of integration, as yet undetermined.

Question:

- What is the arc of quickest descent from a fixed point to a given curve?

- To find the minimizing function, we as before introduce a small variation from $y(x)$, namely

$$y(x) + \varepsilon\eta(x)$$

where ε is a small parameter and $\eta(x)$ is a smooth curve satisfying the BC

$$\eta(a) = 0.$$

- The point of intersection of our small variation $y(x) + \varepsilon\eta(x)$ with the given curve $g(x, y) = 0$ is denoted by (x^*, y^*). We thus have

$$\boxed{g(x^*, y^*) = 0, \qquad y^* = y(x^*) + \varepsilon\eta(x^*)}. \tag{3.5}$$

- We take the derivative of

$$\phi(\varepsilon) = J[y + \varepsilon\eta]$$

with respect to ε, evaluate it $\varepsilon = 0$, and set this equal to zero; that is,

$$\begin{aligned}\frac{d\phi(\varepsilon)}{d\varepsilon} &= \frac{d}{d\varepsilon} J[y + \varepsilon\eta]\Big|_{\varepsilon=0} \\ &= \int_a^{x^*} \left[\frac{\partial F}{\partial y}\eta(x) + \frac{\partial F}{\partial y'}\eta'(x) \right] dx + F(x^*, y^*, (y^*)') \frac{dx^*}{d\varepsilon}\Big|_{\varepsilon=0}\end{aligned}$$

- Since relations (3.5) hold for all ε, we have that the total derivative of $g(x^*, y^*)$ with respect to ε must vanish.
- From (3.5) we therefore obtain, on noting that x^* is a function of ε for any given $\eta(x)$,

$$
\begin{aligned}
0 &= \frac{\partial g}{\partial x^*}\frac{dx^*}{d\varepsilon} + \frac{\partial g}{\partial y^*}\frac{dy^*}{d\varepsilon} \\
&= \frac{\partial g}{\partial x^*}\frac{dx^*}{d\varepsilon} + \frac{\partial g}{\partial y^*}\left[y'(x^*)\frac{dx^*}{d\varepsilon} + \eta(x^*) + \varepsilon\eta'(x^*)\frac{dx^*}{d\varepsilon} \right].
\end{aligned}
$$

- Solving the above equality, with $\varepsilon = 0$, for $\left.\frac{dx^*}{d\varepsilon}\right|_{\varepsilon=0}$ we obtain

$$
\left.\frac{dx^*}{d\varepsilon}\right|_{\varepsilon=0} = -\frac{\eta(x^*)\frac{\partial g}{\partial y^*}}{\frac{\partial g}{\partial x^*} + y'(x^*)\frac{\partial g}{\partial y^*}} \qquad (3.6)
$$

- With the aid of (3.6) and integration by parts we get

$$\int_a^{x^*}\left[\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]\eta dx+\eta(x^*)\left[\frac{\partial F}{\partial y'}-\frac{F\frac{\partial g}{\partial y^*}}{\frac{\partial g}{\partial x^*}+(y^*)'\frac{\partial g}{\partial y^*}}\right]^{x=x^*}=0$$

- Repeating the line of arguments carried out in Problem 3-1 above we conclude that $y=y(x)$ satisfies the Euler-Lagrange equation

$$\frac{\partial F}{\partial y}-\frac{d}{dx}\left[\frac{\partial F}{\partial y'}\right]=0$$

and, in addition to the BC $y(a)=A$, the right-hand end-point condition

$$\boxed{\left[\frac{\partial F}{\partial y'}-\frac{F\frac{\partial g}{\partial y^*}}{\frac{\partial g}{\partial x^*}+(y^*)'\frac{\partial g}{\partial y^*}}\right]^{x=x^*}=0}.$$