Calculus of Variations
Summer Term 2016

Lecture 4

Universität des Saarlandes

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Purpose of Lesson:

- To discuss the generalization of the E-L equation to case of several variables.
- To illustrate the correspondence between the multivariable variational problems and PDEs.

§4. The Case of Several Variables

The Case of Several Variables

In this subsection we consider functionals of the type

$$J[u] = \int\limits_{\Omega} F(x, u(x), Du(x))dx,$$

where

- Ω is a bounded open set in $\mathbb{R}^n$,
- $u : \overline{\Omega} \to \mathbb{R}$ is a function of class C^2,
- $F(x, u, p)$ is a real valued function of class C^2 with respect of all its arguments.

Remark

- The integrand $F(x, u, p)$ is denoted as Lagrangian, or variational integrand, or Lagrange function.

- Consequently, the function
$$\Phi(\varepsilon) := J[u + \varepsilon\eta]$$
is defined for each $\eta \in C_0^{\infty}(\Omega, \mathbb{R})$ and for sufficiently small ε.
- Moreover, Φ is of class C^2 on some interval $(-\varepsilon_0, \varepsilon_0)$.
- We will call its derivative $\Phi'(0)$ at $\varepsilon = 0$ the first variation of J at u in direction of η and denote (sometimes)
$$\delta J[u, \eta] = \Phi'(0).$$
- A stright-forward computation yields
$$\delta J[u, \eta] = \Phi'(0) = \int\limits_{\Omega} \left\{ F_u(x, u, Du)\eta + F_p(x, u, Du) \cdot D\eta \right\} dx$$
$$= \int\limits_{\Omega} \left\{ F_u(x, u, Du)\eta + \sum_{j=1}^{n} \frac{\partial F(x, u, Du)}{\partial p_j} D_j\eta \right\} dx$$

- Involving the integration by parts we end up with

$$
\begin{aligned}
0 &= F_u(x,u,Du) - \sum_{j=1}^{n} \frac{d}{dx_j}\left[\frac{\partial F(x,u,Du)}{\partial p_j}\right] \\
&= F_u - \sum_{j=1}^{n}\left[\frac{\partial^2 F}{\partial p_j \partial x_j} + \frac{\partial^2 F}{\partial p_j \partial u} D_j u + \sum_{k=1}^{n} \frac{\partial^2 F}{\partial p_j \partial p_k} D_k D_j u\right].
\end{aligned}
$$

Correspondence between Variational Integrals and PDEs

Example 4.1 (The Laplace equation)

Consider the *Dirichlet integral* defined by

$$\mathcal{D}[u] := \frac{1}{2}\int\limits_{\Omega} |Du|^2 dx$$

The corresponding Euler-Lagrange equation has the form

$$\Delta u = 0 \quad \text{in} \quad \Omega.$$

Example 4.2 (The Poisson equation)

The integral

$$J[u] = \int\limits_{\Omega} \left[\frac{1}{2} |Du|^2 + f(x)u \right] dx$$

with the Lagrangian

$$F(x, u, p) = \frac{1}{2} |p|^2 + f(x)u$$

has the so-called *The Poisson equation*

$$\Delta u = f(x) \quad \text{in} \quad \Omega$$

as the corresponding E-L equation.

Example 4.3 (The nonlinear Poisson equation)

The nonlinear Poisson equation

$$\Delta u = f(u) \quad \text{in} \quad \Omega$$

is the Euler-Lagrange equation of the integral

$$J[u] = \int_{\Omega} \left\{ \frac{1}{2} |Du|^2 + g(u) \right\} dx$$

where g is a primitive function of f, i.e., $g'(z) = f(z)$.

Example 4.4 (The wave equation)

Interpret $\mathbb{R}^4$ as space-time continuum of points $(x, t) \in \mathbb{R}^3 \times \mathbb{R}$ where x is the position vector in $\mathbb{R}^3$ and t denotes the time.

Let Ω be some bounded domain in $\mathbb{R}^4$, and consider the function $u(x, t)$ of x and t. Then the integral

$$J[u] = \int\limits_{\Omega} \frac{1}{2} \left\{ u_t^2 - |Du|^2 \right\} dxdt$$

has the wave equation

$$\Box u := u_{tt} - \Delta u = 0 \quad \text{in} \quad \Omega$$

as the Euler-Lagrange equation.

Example 4.5 (The minimal surface equation)

The *area functional*

$$\mathcal{A}[u] = \int\limits_{\Omega} \sqrt{1 + |Du|^2} dx$$

for hypersurfaces $z = u(x)$, $x \in \Omega \subset \mathbb{R}^n$, in $\mathbb{R}^{n+1}$ yields the minimal surface equation

$$\operatorname{div} Tu = 0 \quad \text{in} \quad \Omega, \qquad Tu := \frac{Du}{\sqrt{1 + |Du|^2}}$$

as the Euler-Lagrange equation, which we can also write as

$$\sum_{i=1}^{n} D_i \left(\frac{D_i u}{\sqrt{1 + |Du|^2}} \right) = 0.$$

Example 4.6 (Digital Image Processing)

We consider a distorted Black-White-Image which is described by a function $f : \mathbb{R}^2 \supset \Omega \to \mathbb{R}$, here $\Omega = [0, a] \times [0, b]$.

The value $f(x)$ corresponds the darkness of gray color in the image point x (the brighter corresponds the greater value of f). In order to eliminate the interference as much as possible we minimize the functional

$$E_f[u] := \int\limits_{\Omega} \left\{ (u - f)^2 + \alpha |Du|^2 \right\} dx, \qquad \alpha > 0.$$

Here $(f - u)^2$ stands for the difference with the original image and $|Du|$ is a measure of the smoothness of the denoised image. The Integral $E_f[u]$ has the following E-L equation

$$u - \Delta u = f \quad \text{in} \quad \Omega.$$