Calculus of Variations
Summer Term 2016

Lecture 5

Universität des Saarlandes

10. Mai 2016

Purpose of Lesson:

- To consider a class of problems in which the functionals are required to conform with certain restrictions that are added to the usual continuity requirements and possible end-points conditions.

§5. Isoperimetric problems

We now include additional constarints into our minimizations problems:

- Integral constraints of the form

$$\int G(x, y, y')dx = const$$

  e.g., the Isoperimetric problem.
- Holonomic constraints, e.g., $G(x, y) = 0$
- Non-holonomic constraints, e.g., $G(x, y, y') = 0$
- We won't consider inequality constraints until later.

The standard example of a problem with **integral constraints** is *Dido's problem*.

Dido's problem

- This is probably one of the oldest problem in the Calculus of Variations.
- Dido (Carthaginian queen) founded the city of Carthage, in Tunisia.
- According to legend, she arrived at the site with her entourage, a refugee from a power struggle with her brother in Tyre in the Lebanon.
- She asked the locals for as much land as could be bound by a bull's hide.
- She cut the hide into a long thin strip and bounded the maximum possible area with this.

Dido Purchases Land for the Foundation of Carthage. Engraving by Matthäus Merian the Elder, in *Historische Chronica*, Frankfurt a.M., 1630. Dido's people cut the hide of an ox into thin strips and try to enclose a maximal domain.

Dido's problem falls into the class of isoperimetric problems.

- iso- (from same) and perimetric (from perimeter), roughly meaning "same perimeter".
- In general, such problems involve a constraint
  - e.g., the length of the bull's hide strip.
  - But the constraints is not always to fix the perimeter length.
  - Sometimes the constraint does not even involve a length.
  - But the term isoperimetric is still used.

We can write the isoperimetric problems in the following form:

The simple isoperimetric problem:

We are looking for the extremals of the functional

$$J[y] = \int_a^b F(x, y, y')dx \to \min$$

with all the usual conditions (e.g. on ent points, and continuous derivatives) but in addition we must satisfy the extra functional constraint

$$\mathcal{G}[y] = \int_a^b G(x, y, y')dx = L$$

A simplified form of Dido's problem:

Imagine that the two end-points are fixed, along the cost (Carthage was a great sea power), and we wish to encompass the largest possible area inland with a fixed length L.

We can write this problem as maximize the area

$$J[y] = \int_a^b y dx \to \max$$

encompassed by the curve y, such that the curve y has the fixed length L, e.g., as before the length of the curve is

$$\mathcal{G}[y] = \int_a^b \sqrt{1 + y'^2} dx = L$$

subject to the end-pont conditions $y(a) = y(b) = 0$.

A simplified form of Dido's problem:

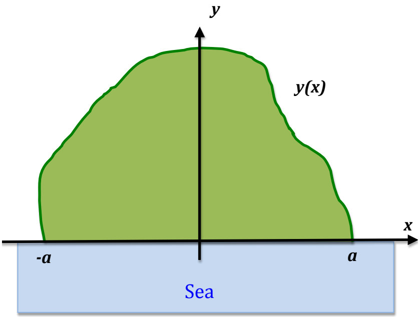
$$J[y] = \int_{-a}^{a} y\,dx \longrightarrow \max$$

subject to

$$G[y] = \int_{-a}^{a} \sqrt{1+(y')^2}\,dx = L$$

and

$$y(-a) = y(a) = 0$$



For simplicity take

$$\boxed{2a < L \leq \pi a}$$

Approach

As before

- we perturb the curve, and consider the first variation
- but we cannot perturb by an arbitrary function $\varepsilon\eta$. because then the constraint

  $$\mathcal{G}[y + \varepsilon\eta] = L$$

  might be violated.
- **solution:** use the same approach as in constrained maximization, e.g. use Lagrange multipliers

Problem

To find the minimum (or maximum) of $f(x)$ for $x \in \mathbb{R}^n$ subject the constraints

$$g_i(x) = 0, \qquad i = 1, \dots, m < n \tag{5.1}$$

- Solution requires Lagrange Multipliers.
- Minimize (ot maximize) a new function (of $m + n$ variables)

$$h(x, \lambda) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x),$$

  where λ_i are the undetermined Lagrange multipliers.
- The constants $\lambda_1, \dots, \lambda_m$ are evaluated by means of the set of equations consisting of (5.1) and

$$\frac{\partial h(x, \lambda)}{\partial x_j} = 0, \qquad j = 1, \dots, n$$

Why Lagrange multipliers?

- Maximize $f(x)$ subject to $g(x) = 0$
$$h(x) = f(x) + \lambda g(x).$$
So $\frac{\partial h}{\partial x_i} = 0$ implies $\frac{\partial f}{\partial x_i} = -\lambda \frac{\partial g}{\partial x_i}$.
- Assume x is an extremum which satisfies the constraint, and consider all of the $x + \Delta x$ in the neighborhood of x that also satisfy the constraint (i.e. $g(x + \Delta x) = g(x) = 0$).
- We also know from Taylor's theorem that
$$g(x + \Delta x) = g(x) + \nabla g(x) \cdot \Delta x + O(\Delta x^2)$$
which implies that for small Δx
$$\nabla g(x) \cdot \Delta x = 0.$$
- If we take $\frac{\partial f}{\partial x_j} = -\lambda \frac{\partial g}{\partial x_j}$ then $\nabla f \cdot \Delta x = 0$.

Lagrange multipliers in functionals

To maximize

$$J[y] = \int_a^b F(x, y, y')dx$$

subject to

$$\mathcal{G}[y] = \int_a^b G(x, y, y')dx = L$$

we instead consider the problem of finding extremals of

$$\mathcal{H}[y] = \int_a^b H(x, y, y')dx = \int_a^b \left\{ F(x, y, y') + \lambda G(x, y, y') \right\} dx$$

The Euler-Lagrange equations

The Euler-Lagrange equations become

$$\boxed{\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) = 0}$$

where $H = F + \lambda G$, and λ is the unknown Lagrange multiplier.

Example 5.1 (Simple Dido's problem)

$$\mathcal{H}[y] = \int_{-a}^{a} \left(y + \lambda\sqrt{1 + (y')^2} \right) dx$$

so

$$\frac{\partial H}{\partial y} = 1$$

$$\frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) = \frac{d}{dx}\left(\frac{\lambda y'}{\sqrt{1+(y')^2}}\right)$$

and the Euler-Lagrange equation is

$$\frac{d}{dx}\frac{\lambda y'}{\sqrt{1+(y')^2}} = 1$$

Example 5.1 (Simple Dido's problem)

- Integrating with respect to x we get

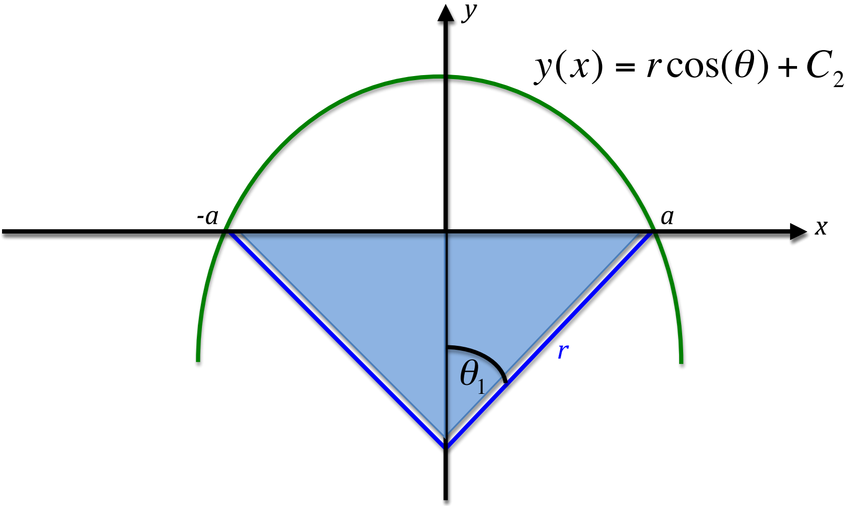
$$\begin{aligned} x + C_1 &= \lambda \sin(\theta) \\ y &= -\lambda \cos(\theta) + C_2 \end{aligned}$$

  where λ, C_1 and C_2 are determined by the two end-points, and the length of the curve L.

- We may draw a sketch of the solution, and clearly we can identify $-\lambda = r$ the radius of a circle, of which our region is a segment.

- Note we deliberately started with

$$2a < L \leqslant \pi a.$$


y
$y(x) = r\cos(\theta) + C_2$
-a
a
x
r
θ_1

Example 5.1 (Simple Dido's problem)

- We can see that the arc length of the enclosing curve will be

$$L = 2\theta_1 r$$

and the the value on the right-end determines that

$$r = \frac{a}{\sin(\theta_1)}$$

- Therefore, we have

$$L = \frac{2a\theta_1}{\sin(\theta_1)}$$

from which we may determine θ_1.

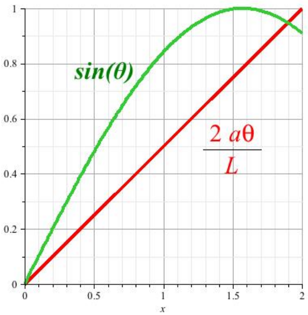
Example 5.1 (Simple Dido's problem)

- Since we determine θ_1 from

$$\sin(\theta_1) = \frac{2a}{L}\theta_1$$

we may compute

$$r = \frac{a}{\sin(\theta_1)}.$$



Example 5.1 (Simple Dido's problem)

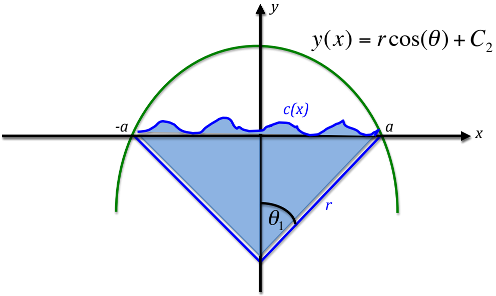
- From the conditions $y(\pm a) = 0$ it follows that

$$C_2 = -a\cot(\theta_1).$$

- The maximum possible area bounded by a curve of fixed length is a circle. So the city of Carthage is circular in shape.
- The story of Carthage isn't quite true (see picture below).

What effect would a realistic coastline have?

- Coast $c(x)$.



- Area $= \int_{-a}^{a} (y - c)\, dx$
- Note that c doesn't depend on y or y', so the Euler-Lagrange equations are unchanged, provided $c(x) < y(x)$ for the extremal.

What effect would a realistic coastline have?

- If the condition $c(x) < y(x)$ is not satisfied then the area integral includes negative components, so the problem we are maximizing is not really Dido's problem any more (she can't own negative areas).
- We really want to maximize

$$\text{Area} = \int_a^b [y - c]^+ \, dx$$

where

$$[x]^+ = \begin{cases} x, & \text{for } x > 0 \\ 0, & \text{otherwise} \end{cases}$$

- Note that the function $[x]^+$ does not have a derivative at $x = 0$.