Universität des Saarlandes
Fachrichtung 6.1 – Mathematik
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Calculus of Variations (Summer Term 2017)
Exam-Part W

Problem E.1 (10 Points)

Find the coordinates of the point(s) nearest the origin on the surface $xyz = R^3$, for $x, y, z \geq 0$.

Show (using the transversal conditions and the Euler-Lagrange equations) that if we were to draw a line between this point and the origin, it would be a transversal of minimum length between the origin and the surface.

Problem E.2 (10 Points)

Find the curves on which the following functional can attain an extremum

$$J[y] = \int_0^{10} (y')^3 dx, \qquad y(0) = 0, \quad y(10) = 0$$

subject to the condition that the admissible curves cannot pass inside the area bounded by the circle $(x-5)^2 + y^2 = 9$.

Problem E.3 (10 Points)

Consider the functional

$$J[y] = \int_0^2 \left[(y'^2 + y^2)(y-x)^2 - \frac{4}{3}y^3 + 2xy^2 \right] dx$$

$$y(0) = 0, \qquad y(2) = e$$

a) Find the corresponding Euler-Lagrange equation and find out at what points in $[0, 2]$ it must hold.

b) Show that $y = x$ and $y = \alpha e^x$ are solutions of the equation found in part a).

c) Find all possible corner points for the extremals of J.

d) Find a possible extremal for our problem.

Problem E.4 (10 Points)

Find an approximate solution of the nonlinear equation

$$y'' = \frac{3}{2}y^2$$

that satisfies the conditions $y(0) = 4$ and $y(1) = 1$.

Hints:

1. *Find a variational problem that corresponds to the above BVP.*
2. *Apply the Ritz method.*

Problem E.5 (10 Points)

Solve the following optimal control problems with the help of PMP:

(a) Minimize

$$F(x) = \int_0^{10} x^2 dt$$

subject to

$$|\ddot{x}| \leqslant 1, \text{and } x(0) = 1.$$

(b) Minimize T subject to

$$\int_0^T \ddot{x}^2 dt = 4$$

and

$$x(0) = 1, \text{and } \dot{x}(0) = 1, \text{and } \dot{x}(T) = -2.$$

Deadline for submission: Monday, September 18, 2017