Calculus of Variations
Summer Term 2017

Lecture 11

Universität des Saarlandes

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Purpose of Lesson:

- To introduce the notion of a broken extremal
- To discuss the properties of broken extremals

§10. Broken extremals

- Until now we mostly stude the extremals curves with at least two well-defined derivatives.
- Obviously this is not always true.
- Broken extremals are continuous extremals for which the gradient has a discontinuty at one or more points.
- If a variational problem has a smooth extremal (That therefore satisfies the Euler-Lagrange equations), this will be better than a broken one.
- But some problems don't admit smooth extremals.

Example 10.1

Find $y(x)$ to minimize

$$J[y] = \int_{-1}^{1} y^2(1 - y')^2 dx$$

subject to $y(-1) = 0$ and $y(1) = 1$.

Example 10.1 (cont.)

- There is no explicit x dependence inside the integral, so we can find

$$\begin{aligned} F - y' F_{y'} &= c_1 = const \\ y^2(1-y')^2 + 2y'y^2(1-y') &= c_1 \\ y^2(1-y')\left[1+y'\right] &= c_1 \\ y^2\left[1-y'^2\right] &= c_1 \end{aligned}$$

- If $c_1 = 0$ we get the singular solutions

$$y = 0 \qquad \text{or} \qquad y = \pm x + B.$$

Neither of these satisfies both end-ponts conditions $y(-1) = 0$ and $y(1) = 1$, so $c_1 \neq 0$ (we think).

Example 10.1 (cont.)

- Given $c_1 \neq 0$

$$
\begin{aligned}
y^2 \left[1 - y'^2\right] &= c_1 \\
y'^2 &= \frac{y^2 - c_1}{y^2} \\
\frac{dy}{dx} &= \pm\frac{1}{y}\sqrt{y^2 - c_1} \\
dx &= \pm\frac{y}{\sqrt{y^2 - c_1}}dy \\
x &= \pm\sqrt{y^2 - c_1} + c_2 \\
(x - c_2)^2 &= y^2 - c_1
\end{aligned}
$$

- The solution is a rectangular hyperbola.

Example 10.1 (cont.)

- Using the end-points conditions we find c_1 and c_2 from

$$(x - c_2)^2 = y^2 - c_1 .$$

$$\begin{aligned} y(-1) = 0 \quad &\Rightarrow \quad (-1 - c_2)^2 = -c_1 \\ y(1) = 1 \quad &\Rightarrow \quad (1 - c_2)^2 = 1 - c_1 \end{aligned}$$

- Addition of these two equations gives

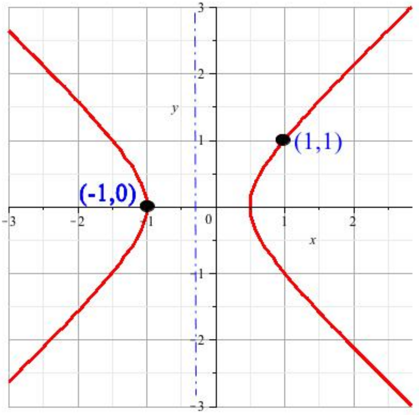
$$(1 - c_2)^2 = 1 + (1 + c_2)^2$$

which has solution $c_2 = -1/4$, and so $c_1 = -9/16$

$$y^2 = (x + 1/4)^2 - 9/16.$$

Example 10.1 (cont.)

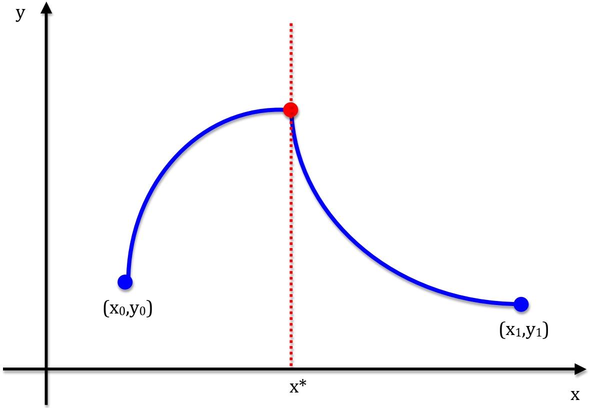
- The end-points are on opposite branches of the hyperbola!



- There is NO smooth extremal curve that connects $(-1, 0)$ and $(1, 1)$.

- Sometimes there is no smooth extremal.
- We must seek a broken extremal.
- Still want a continuous extremal.
- What should we do?
    - Previous smothness results suggest that we should use a smooth extremal when we can, and so we will try to minimize the number of corners.
    - We'll start by looking for curves with one corner.
    - But can we apply the Euler-Lagrange equations?

- If we have an extremal like this, can we use the Euler-Lagrange equations?



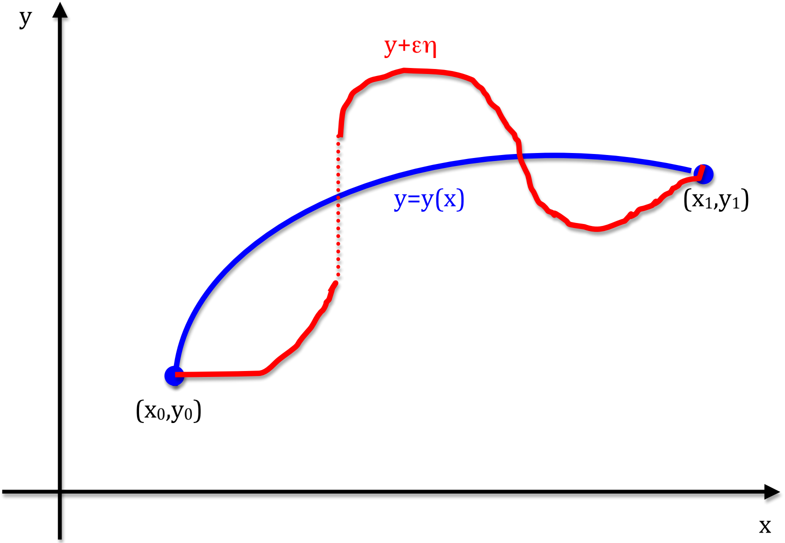
Smoothness theorem

Theorem 10.1

If the smooth curve $y(x)$ gives an extremal of a functional $J[y]$ over the class of all admissible curves in some ε neighborhood of y, then $y(x)$ also gives an extremal of a functional $J[y]$ over the class of all piecewise smooth curves in the same neighborhood.

Meaning:

We can extend our results to piecewise smooth curves (where a smooth result exists), not just curves with 2 continuous derivatives.


y
y+εη
y=y(x)
(x0,y0)
(x1,y1)
x

Proof Sketch

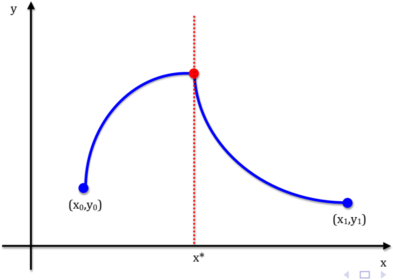
- The theorem assumes that there exists a smooth extremal (in this case a minimum for the purpose of illustration) y. Then for any other smooth curve $\hat{y} \in B_\varepsilon(y)$ we know $J[\hat{y}] > J[y]$.
- Assume for the moment that for a piecewise smooth function $\tilde{y} \in B_\varepsilon(y)$ we have $J[\tilde{y}] < J[y]$. We can approximate $\tilde{y}$ by a smooth curve $\hat{y}_\delta \in B_\varepsilon(y)$ by rounding off the edges of the discontinuity.
- Given that we can approximate the curve $\tilde{y}$ arbitrarily closely by a smooth curve $\hat{y}_\delta$, for which we already know $J[\hat{y}_\delta] > J[y]$. We get a contradiction with $J[\tilde{y}] < J[y]$, and so no such alternative extremal can exist.

So What We Do?

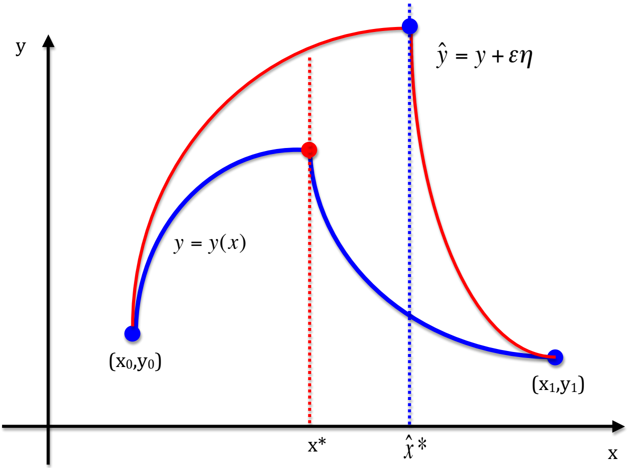
- Break the functional into two parts:

$$J[y] = J_1[y] + J_2[y] = \int\limits_{x_0}^{x^*} F(x, y_1, y_1')dx + \int\limits_{x^*}^{x_1} F(x, y_2, y_2')dx$$

- We require y to have two continuous derivatives everywhere except at x^*, and $y_1(x^*) = y_2(x^*)$.



Possible Perturbations:



The location of the "corner" can also be perturbed

The First Variation: part 1

- We get the first component of the first variation by considering a problem with only one fixed end-point, and allowing x^* to vary, so that

$$0 = \left.\frac{d\phi_1(\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \int\limits_{x_0}^{\hat{x}^*} F(x, y_1 + \varepsilon\eta, y_1' + \varepsilon\eta')dx$$

$$= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \int\limits_{x_0}^{x^*+\varepsilon X} F(x, y_1 + \varepsilon\eta, y_1' + \varepsilon\eta')dx$$

$$= \left. XF(x, y_1, y_1')\right|_{x=x^*} + \left. F_{y_1'}\eta\right|_{x=x^*} + \int\limits_{x_0}^{x^*} \left(F_{y_1} - \frac{d}{dx}F_{y_1'}\right)\eta dx$$

The First Variation: part 1

- The perturbed point $(\hat{x}^*, \hat{y}^*)$ and perturbed function η must satisfy certain conditions to be compatible.
- Remember that

$$\begin{aligned} \hat{x}^* &= x^* + \varepsilon X \\ \hat{y}^* &= y^* + \varepsilon Y \end{aligned}$$

- Notice that

$$\hat{y}^* = y(x^* + \varepsilon X) + \varepsilon \eta(x^* + \varepsilon X).$$

- From Taylor's theorem, for small ε

$$\begin{aligned} y(x^* + \varepsilon X) &= y(x^*) + \varepsilon X y'(x^*) + O(\varepsilon^2) \\ &= y^* + \varepsilon X y'(x^*) + O(\varepsilon^2) \\ \varepsilon \eta(x^* + \varepsilon X) &= \varepsilon \eta(x^*) + O(\varepsilon^2) \end{aligned}$$

The First Variation: part 1

- So

$$\begin{aligned} y^* + \varepsilon Y &= y^* + \varepsilon X y'(x^*) + \varepsilon\eta(x^*) + O(\varepsilon^2) \\ \varepsilon Y &= \varepsilon X y'(x^*) + \varepsilon\eta(x^*) + O(\varepsilon^2) \\ \eta(x^*) &= Y - X y'(x^*) + O(\varepsilon) \end{aligned}$$

- Thus, we have

$$\boxed{\eta(x^*) = Y - X y'(x^*) + O(\varepsilon)} \tag{10.1}$$

The First Variation: part 1

- Substituting the compatibility constraint (10.1) into the our first variation we get

$$0 = \left[XF + F_{y_1'}\eta\right]_{x=x^*} + \int\limits_{x_0}^{x^*} \left(F_{y_1} - \frac{d}{dx}F_{y_1'}\right)\eta dx$$

$$= XF\big|_{x=x^*} + \left[Y - Xy_1'(x^*)\right]F_{y_1'}\big|_{x=x^*} + \int\limits_{x_0}^{x^*} \left(F_{y_1} - \frac{d}{dx}F_{y_1'}\right)\eta dx$$

$$= X\left[F - y_1'F_{y_1'}\right]_{x=x^*} + YF_{y_1'}\big|_{x=x^*} + \int\limits_{x_0}^{x^*} \left(F_{y_1} - \frac{d}{dx}F_{y_1'}\right)\eta dx$$

The First Variation: part 1

- So, we get an integral term which results in the E-L equation, plus the additional constraint

$$X\left[F - y_1' F_{y_1'}\right]_{x=x^*} + Y F_{y_1'}\Big|_{x=x^*} = 0 \tag{10.2}$$

The First Variation: part 2

- Note that, for the second component of the First Variation we get a similar extra term, e.g.

$$-X\left[F - y_2' F_{y_2'}\right]_{x=x^*} - Y F_{y_2'}\Big|_{x=x^*} = 0. \qquad (10.3)$$

- The sign is reversed because it corresponds to the x_0 term (as opposed to the x_1 term for δJ_1).

- The combined First Variation (minus the terms that result from the Euler-Lagrange equation which must be zero) is

$$X\left[F - y_1' F_{y_1'}\right]_{x=x^*} + Y F_{y_1'}\Big|_{x=x^*} - X\left[F - y_2' F_{y_2'}\right]_{x=x^*} - Y F_{y_2'}\Big|_{x=x^*} = 0.$$

Conditions

- We rearrange to give

$$0 = X\left\{\left[F(x, y_1, y_1') - y_1' F_{y_1'}\right] - \left[F(x, y_2, y_2') - y_2' F_{y_2'}\right]\right\}_{x=x^*} + Y\left\{F_{y_1'} - F_{y_2'}\right\}_{x=x^*}.$$

- Note that the point of discontinuity may vary freely, so we may independently vary X and Y or set one or both to zero. Hence, we can separate the condition to get two conditions

$$\left[F(x, y_1, y_1') - y_1' F_{y_1'} - F(x, y_2, y_2') + y_2' F_{y_2'}\right]_{x=x^*} = 0$$
$$\left\{F_{y_1'} - F_{y_2'}\right\}_{x=x^*} = 0$$

Weierstrass-Erdman

- We can write the conditions as

$$\Big[F(x, y_1, y_1') - y_1' F_{y_1'}\Big]_{x=x^*} = \Big[F(x, y_2, y_2') - y_2' F_{y_2'}\Big]_{x=x^*}$$

$$F_{y_1'}\Big|_{x=x^*} = F_{y_2'}\Big|_{x=x^*}$$

Called the Weierstrass-Erdmann Corner Conditions.

- Rather than separating y into y_1 and y_2 we may write the corner conditions in terms of limits from the left and right, e.g.

$$\Big[F - y' F_{y'}\Big]_{x=x^{*-}} = \Big[F - y' F_{y'}\Big]_{x=x^{*+}}$$

$$F_{y'}\Big|_{x=x^{*-}} = F_{y'}\Big|_{x=x^{*+}}$$

Solution

So the broken extremal solution must satisfy

- The Euler-Lagrange equations
- The Weierstrass-Erdmann Corner Conditions

$$\left[F - y'F_{y'}\right]_{x=x^{*-}} = \left[F - y'F_{y'}\right]_{x=x^{*+}}$$

$$F_{y'}\Big|_{x=x^{*-}} = F_{y'}\Big|_{x=x^{*+}}$$

must hold at any ”corner”.

Example 10.1 (cont. ii)

Find $y(x)$ to minimize

$$J[y] = \int_{-1}^{1} y^2(1 - y')^2 dx$$

subject to $y(-1) = 0$ and $y(1) = 1$.

Example 10.1 (cont. ii)

- In the example considered

$$F - y' F_{y'} = y^2 \left(1 - y'^2\right)$$
$$F_{y'} = -2y^2 \left(1 - y'\right)$$

- Remember that $y = 0$ and $y = x + A$ are valid solutions to the Euler-Lagrange equations, and that for both of these solutions

$$F_{y'} = F - y' F_{y'} = 0,$$

so we can put a ”corner” where needed.

Example 10.1 (cont. ii)

- The solution must also satisfy the end-point conditions, so $y(-1) = 0$ and $y(1) = 1$, and therefore, as valid solution has $x^* = 0$ and

$$\begin{aligned} y_1 &= 0 \quad \text{for} \quad x \in [-1, x^*] \\ y_2 &= x \quad \text{for} \quad x \in [x^*, 1] \end{aligned}$$