Calculus of Variations
Summer Term 2017

Lecture 12

Universität des Saarlandes

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Purpose of Lesson:

- To discuss numerical solutions of the variational problems
- To introduce Euler's Finite Difference Method and Ritz's Method.

§11. Numerical Solutions

Numerical Solutions:

The Euler-Lagrange equations may be hard to solve.

Natural response is to find numerical methods.

1. Numerical solution of the Euler-Lagrange equations
   - We won't consider these here (see other courses)
2. Euler's finite difference method
3. Ritz (Rayleigh-Ritz)
   - In $2D$: Kantorovich's method

Euler's Finite Difference Method

- We can approximate our function (and hence the integral) onto a finite grid.
- In this case, the problem reduces to a standard multivariable maximization (or minimization) problem, and we find the solution by setting the derivatives to zero.
- In the limit as the grid gets finer, this approximates the Euler-Lagrange equations.

Numerical approximation of integrals:

- use an arbitrary set of mesh points

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

- approximate

$$y'(x_i) = \frac{y_{i+1} - y_i}{x_{i+1} - x_i} = \frac{\Delta y_i}{\Delta x_i}$$

- rectangle rule

$$J[y] = \int_a^b F(x, y, y')dx \simeq \sum_{i=0}^{n-1} F\left(x_i, y_i, \frac{\Delta y_i}{\Delta x_i}\right) \Delta x_i = \widehat{J}[\mathbf{y}]$$

$\widehat{J}[\cdot]$ is a function of the vector $\mathbf{y} = (y_0, y_1, y_2, \dots, y_n)$.

Finite Difference Method (FDM)

- Treat this as a maximization of a function of $n+1$ variables, so that we require

$$\frac{\partial \widetilde{J}}{\partial y_i} = 0$$

for all $i = 0, 1, 2, \ldots, n$.

- Typically use uniform grid so

$$\Delta x_i = \Delta x = \frac{b-a}{n}.$$

Example 11.1

Find extremals for

$$J[y] = \int_0^1 \left[\frac{1}{2}y'^2 + \frac{1}{2}y^2 - y\right] dx$$

with $y(0) = 0$ and $y(1) = 0$.

The Euler-Lagrange equation $y'' - y = -1$.

Example 11.1 (direct solution)

- E-L equation: $y'' - y = -1$
- Solution to homogeneous equation $y'' - y = 0$ is given by $e^{\lambda x}$ giving characteristic equation

$$\lambda^2 - 1 = 0,$$

so $\boxed{\lambda = \pm 1}$

- Particular solution $y = 1$.
- Final solution is

$$\boxed{y(x) = Ae^{x} + Be^{-x} + 1}.$$

Example 11.1 (direct solution)

- The boundary conditions $y(0) = y(1) = 0$ constrain

$$A + B = -1$$
$$Ae + Be^{-1} = -1$$

so $\boxed{A = \frac{1-e}{e^2-1}}$ and $\boxed{B = \frac{e-e^2}{e^2-1}}$.

- Then the exact solution to the extremal problem is

$$\boxed{y(x) = \frac{1-e}{e^2-1}e^x + \frac{e-e^2}{e^2-1}e^{-x} + 1}.$$

Example 11.1 (Euler's FDM)

Find extremals for

$$J[y] = \int_0^1 \left[\frac{1}{2}y'^2 + \frac{1}{2}y^2 - y\right] dx$$

Euler's FDM:

- Take the grid $x_i = i/n$, for $i = 0, 1, \ldots, n$ so
  - end points $y_0 = 0$ and $y_n = 0$
  - $\Delta x = 1/n$ and $\Delta y_i = y_{i+1} - y_i$.
- So
  - $y_i' = \Delta y_i / \Delta x = n(y_{i+1} - y_i)$
  - and
$$y_i'^2 = n^2 \left(y_i^2 - 2y_i y_{i+1} + y_{i+1}^2\right).$$

Example 11.1 (Euler's FDM)

Find extremals for

$$J[y] = \int_0^1 \left[\frac{1}{2}y'^2 + \frac{1}{2}y^2 - y\right] dx$$

Its FDM approximation is

$$\begin{aligned}\widetilde{J}[\mathbf{y}] &= \sum_{i=0}^{n-1} F(x_i, y_i, y_i')dx \\ &= \sum_{i=0}^{n-1} \frac{1}{2}n^2 \left(y_i^2 - 2y_i y_{i+1} + y_{i+1}^2\right) \Delta x + \left(y_i^2/2 - y_i\right) \Delta x \\ &= \sum_{i=0}^{n-1} \frac{1}{2}n \left(y_i^2 - 2y_i y_{i+1} + y_{i+1}^2\right) + \frac{y_i^2/2 - y_i}{n}.\end{aligned}$$

Example 11.1 (end-conditions)

- We know the end conditions $y(0) = y(1) = 0$, which imply that

$$y_0 = y_n = 0.$$

- Include them into the objective using Lagrange multipliers

$$\mathcal{H}[\mathbf{y}] = \sum_{i=0}^{n-1} \frac{1}{2} n \left(y_i^2 - 2y_i y_{i+1} + y_{i+1}^2 \right) + \frac{y_i^2/2 - y_i}{n} + \lambda_0 y_0 + \lambda_n y_n.$$

Example 11.1 (Euler's FDM)

- Taking derivatives, note that y_i only appears in two terms of the FDM approximation

$$\mathcal{H}[\mathbf{y}] = \sum_{i=0}^{n-1} \frac{1}{2} n \left(y_i^2 - 2 y_i y_{i+1} + y_{i+1}^2 \right) + \frac{y_i^2/2 - y_i}{n} + \lambda_0 y_0 + \lambda_n y_n$$

$$\frac{\partial \mathcal{H}[\mathbf{y}]}{\partial y_i} = \begin{cases} n(y_0 - y_1) + \dfrac{y_0 - 1}{n} + \lambda_0 & \text{for} \quad i = 0 \\ n(2y_i - y_{i+1} - y_{i-1}) + y_i/n - 1/n & \text{for } i = 1, \dots, n-1 \\ n(y_n - y_{n-1}) + \lambda_n & \text{for} \quad i = n \end{cases}$$

- We need to set the derivatives to all be zero, so we now have $n+3$ linear equations, including $y_0 = y_n = 0$, and $n+3$ variables including the two Lagrange multipliers.

- We can solve this system numerically using, e.g., Maple.

Example 11.1 (Euler's FDM)

Example: $n = 4$, solve

$$A\mathbf{z} = \mathbf{b}$$

where

$$A = \begin{pmatrix} 1.00 & & & & & & \\ 4.25 & -4.00 & & & & 1 & \\ -4.00 & 8.25 & -4.00 & & & & \\ & -4.00 & 8.25 & -4.00 & & & \\ & & -4.00 & 8.25 & -4.00 & & \\ & & & -4.00 & 4.00 & & 1.00 \\ & & & & 1.00 & & \end{pmatrix}$$

and

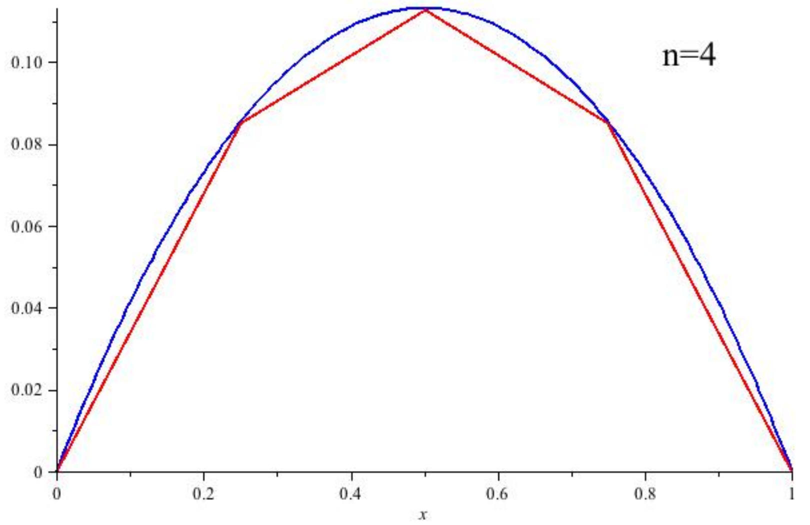
$$\mathbf{b} = (0.00,\ 0.25,\ 0.25,\ 0.25,\ 0.25,\ 0.00,\ 0.00)^T$$

Example 11.1 (Euler's FDM)

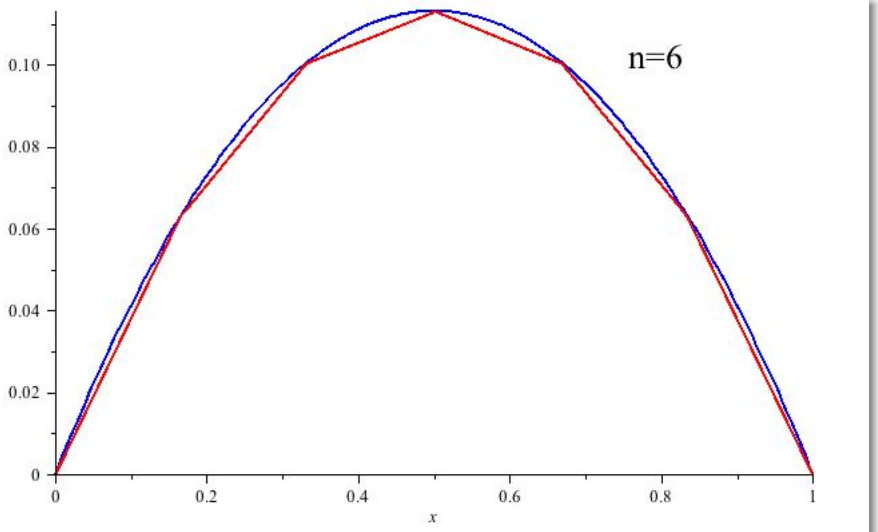
- First $n+1$ terms of $\mathbf{z}$ give $\mathbf{y}$
- Last two terms of $\mathbf{z}$ give the Lagrange multipliers λ_0 and λ_n.
- Solving the system we get for $n=4$

$$y_1 = y_3 = 0.08492201040, \qquad y_2 = 0.1126516464$$

Example 11.1 (results)



Example 11.1 (results)



Convergence of Euler's FDM

$$\widehat{J}[\mathbf{y}] = \sum_{i=0}^{n-1} F\left(x_i, y_i, \frac{\Delta y_i}{\Delta x_i}\right)\Delta x \quad \text{and} \quad \Delta y_i = y_{i+1} - y_i$$

Only two terms in the sum involve y_i, so

$$\begin{aligned}
\frac{\partial \widehat{J}}{\partial y_i} &= \frac{\partial}{\partial y_i} F\left(x_{i-1}, y_{i-1}, \frac{\Delta y_{i-1}}{\Delta x}\right) + \frac{\partial}{\partial y_i} F\left(x_i, y_i, \frac{\Delta y_i}{\Delta x}\right) \\
&= \frac{1}{\Delta x}\frac{\partial F}{\partial y_i'}\left(x_{i-1}, y_{i-1}, \frac{\Delta y_{i-1}}{\Delta x}\right) \\
&\quad + \frac{\partial F}{\partial y_i}\left(x_i, y_i, \frac{\Delta y_i}{\Delta x}\right) - \frac{1}{\Delta x}\frac{\partial F}{\partial y_i'}\left(x_i, y_i, \frac{\Delta y_i}{\Delta x}\right) \\
&= \frac{\partial F}{\partial y_i}(x_i, y_i, y_i') - \frac{\frac{\partial F}{\partial y_i'}\left(x_i, y_i, \frac{\Delta y_i}{\Delta x}\right) - \frac{\partial F}{\partial y_i'}\left(x_{i-1}, y_{i-1}, \frac{\Delta y_{i-1}}{\Delta x}\right)}{\Delta x}
\end{aligned}$$

Convergence of Euler's FDM

$$\frac{\partial \widehat{J}}{\partial y_i} = \frac{\partial F}{\partial y_i}(x_i, y_i, y_i') - \frac{\frac{\partial F}{\partial y_i'}\left(x_i, y_i, \frac{\Delta y_i}{\Delta x}\right) - \frac{\partial F}{\partial y_i'}\left(x_{i-1}, y_{i-1}, \frac{\Delta y_{i-1}}{\Delta x}\right)}{\Delta x} = 0.$$

In limit $n \to \infty$, then $\Delta x \to 0$, and so we get

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

which are the Euler-Lagrange equations.

- i.e., the finite difference solution converges to the solution of the Euler-Lagrange equations.

Remarks

- There are lots of ways to improve Euler's FDM
  - use a better method of numerical quadrature (integration)
    - trapezoidal rule
    - Simpson's rule
    - Romberg's method
  - use a non-uniform grid
    - make it finer where there is more variation
- We can use a different approach that can be even better.

Ritz's Method

- In Ritz's method (called Kantorovich's method where there is more than one independent variable), we approximate our functions (the extremal in particular) using a family of simple functions.
- Again we can reduce the problem into a standard multivariable maximization problem, but now we seek coefficients for our approximation.

Assume we can approximate $y(x)$ by

$$y(x) = \phi_0(x) + c_1\phi_1(x) + c_2\phi_2(x) + \cdots + c_n\phi_n(x)$$

where we choose a convenient set of functions $\phi_j(x)$ and find the values of c_j which produce an extremal.

For fixed end-points problem:

- Choose $\phi_0(x)$ to satisfy the end conditions.
- Then $\phi_j(x_0) = \phi_j(x_1) = 0$ for $j = 1, 2, \ldots, n$

The ϕ can be chosen from standard sets of functions, e.g. power series, trigonometric functions, Bessel's functions, etc. (but must be linearly independent).

- Select $\{\phi_j\}_{j=0}^{n}$
- Approximate

$$y_n(x) = \phi_0(x) + c_1\phi_1(x) + c_2\phi_2(x) + \cdots + c_n\phi_n(x)$$

- Approximate $J[y] \simeq J[y_n] = \int\limits_{x_0}^{x_1} F(x, y_n, y_n')dx$.
- Integrate to get $J[y_n] = J_n(c_1, c_2, \ldots, c_n)$.
- J_n is a known function of n variables, so we can maximize (or minimize) it as usual by

$$\frac{\partial J_n}{\partial c_i} = 0$$

  for all $i = 1, 2, \ldots, n$.

- Assume the extremal of interest is a minimum, then for the extremal

$$J[y] < J[\hat{y}]$$

for all $\hat{y}$ within the neighborhood of y.

- Assume our approximating function y_n is close enough to be in that neighborhood, then

$$J[y] \leqslant J[y_n] = J_n[\mathbf{c}]$$

so the approximation provides an upper bound on the minimum $J[y]$.

- Another way to think about it is that we optimize on a smaller set of possible functions y, so we can't get quite as good a minimum.

Application of the Ritz method: example 11.1a

Find extremals for

$$J[y] = \int_0^1 \left[\frac{1}{2}y'^2 + \frac{1}{2}y^2 - y\right] dx$$

with $y(0) = 0$ and $y(1) = 0$.

The Euler-Lagrange equation $y'' - y = 1$, but we shall bypass the Euler-Lagrange equation to use Ritz's method.

$$y_n(x) = \phi_0(x) + \sum_{i=1}^{n} c_i \phi_i(x)$$

where we take $\phi_0(x) = 0$ and $\phi_i(x) = x^i (1-x)^i$.

Example 11.1a

- Simple approximation $y_1 = c_1\phi_1(x)$ we get

$$J_1[c_1] = J[y_1] = \int\limits_0^1 \left[\frac{1}{2}c_1^2\phi_1'^2 + \frac{1}{2}c_1^2\phi_1^2 - c_1\phi_1\right] dx.$$

- Now $\phi_1(x) = x(1-x)$ so $\phi_1' = 1 - 2x$, and

$$\begin{aligned} J_1[c_1] &= \int\limits_0^1 \left[\frac{c_1^2}{2}(1-2x)^2 + \frac{c_1^2}{2}x^2(1-x)^2 - c_1 x(1-x)\right] dx \\ &= \frac{c_1^2}{2}\int\limits_0^1 \left[1 - 4x + 5x^2 - 2x^3 + x^4\right] dx + c_1\int\limits_0^1 \left[-x + x^2\right] dx \\ &= \frac{11c_1^2}{60} - \frac{c_1}{6}. \end{aligned}$$

Example 11.1a

- We solve for c_1 by setting

$$\frac{dJ_1}{dc_1} = \frac{11c_1}{30} - \frac{1}{6} = 0$$

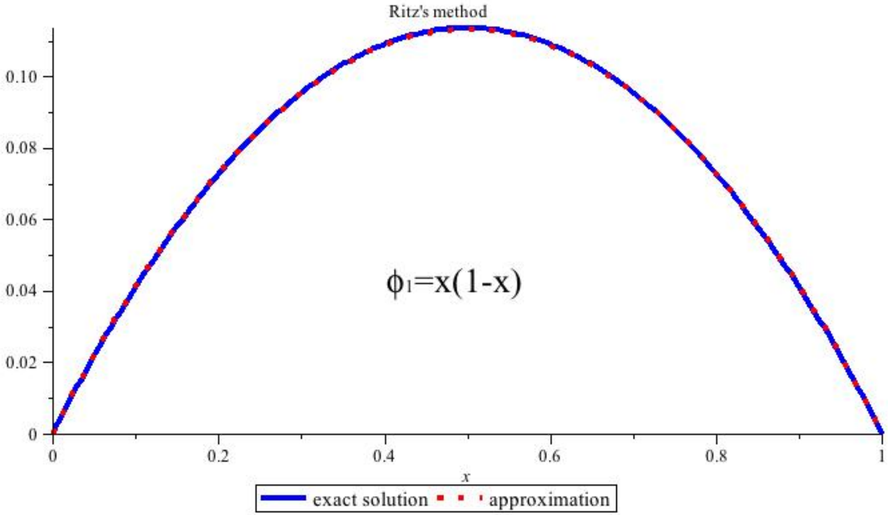
  to get $c_1 = 5/11$, so the approximate extremal is

$$y_1(x) = \frac{5}{11}x(1-x).$$

- The value of the approximate functional at this point is

$$J_1[5/11] = \frac{11c_1^2}{60} - \frac{c_1}{6} = -0.37879$$

  which is an upper bound on the true value of the functional on the extremal.


Ritz's method
0.10
0.08
0.06
0.04
0.02
0
0
0.2
0.4
0.6
0.8
1
x
ϕ1=x(1-x)
exact solution
approximation

Example 11.1a (alternate approach)

- Choose $\phi_1(x) = \sin(\pi x)$ (use the first element of a trigonometric series to approximate y).

- Then, $\phi_1'(x) = \pi\cos(\pi x)$, and so the functional is

$$J_1[c_1] = J[c_1\phi_1] = \int_0^1 \left[\frac{1}{2}c_1^2\phi_1'^2 + \frac{1}{2}c_1^2\phi_1^2 - c_1\phi_1\right] dx$$

$$= \int_0^1 \left[\frac{c_1^2\pi^2}{2}\cos^2(\pi x) + \frac{c_1^2}{2}\sin^2(\pi x) - c_1\sin(\pi x)\right] dx.$$

- Observe that $\int_0^1 \cos^2(\pi x)dx = \int_0^1 \sin^2(\pi x)dx = 1/2$, and $\int_0^1 \sin(\pi x)dx = \left[-\frac{1}{\pi}\cos(\pi x)\right]_0^1 = -2/\pi$.

Example 11.1a (alternate approach)

- So

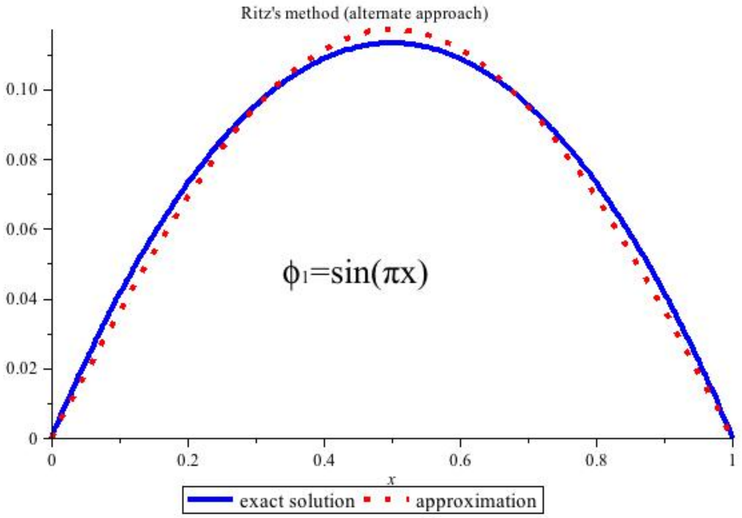
$$J_1[c_1] = \frac{c_1^2}{4}\left[\pi^2 + 1\right] - \frac{2}{\pi}c_1.$$

- Once again we solve for c_1 by setting

$$\frac{dJ_1}{dc_1} = \frac{c_1}{2}\left[\pi^1 + 1\right] - \frac{2}{\pi} = 0$$

to get $c_1 = \frac{4}{\pi(\pi^2+1)}$, so the approximate extremal is

$$y_1(x) = \frac{4}{\pi(\pi^2 + 1)} \sin(\pi x).$$


Ritz's method (alternate approach)
0.10
0.08
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0.04
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0.2
0.4
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0.8
1
x
$\phi_1 = \sin(\pi x)$
exact solution
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