# Calculus of Variations
# Summer Term 2017

Lecture 13

Universität des Saarlandes

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## Purpose of Lesson:

- The Ritz method applied to the catenary gives additional insights.
- Kantorovich's method generalizes Ritz to $2D$ functions.
- To consider optimal control examples
- To introduce a terminology.

# Ritz and Catenary:

## Example 11.2 (the catenary, again)

The functional of interest (the potential energy) is

$$J_p[y] = mg \int_{x_0}^{x_1} y\sqrt{1 + y'^2}\,dx.$$

- Take symmetric problem with fixed end points $y(-1) = a$ and $y(1) = a$.
  We know that the solution looks like

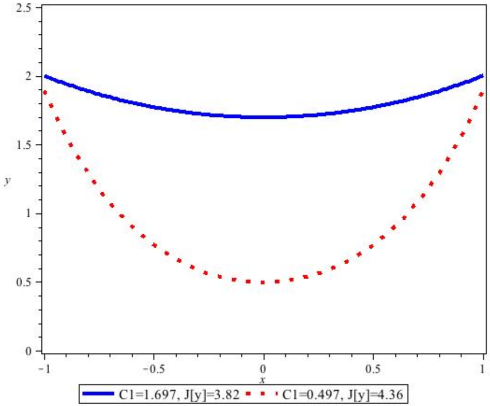
  $$y(x) = c_1 \cosh\left(\frac{x}{c_1}\right)$$

  where $c_1$ is chosen to match the end points.

## Example 11.2 (the catenary, again-2)

$y(1) = 2$ gives $c_1 = 0.47$ or $c_1 = 1.697$

- Are they both local minima?



## Example 11.2 (Ritz and the Catenary)

- Lets try approximating the curve by a polynomial

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots$$

- Note that symmetry of problem implies $y$ is an even function, and hence the odd terms

$$a_1 = a_3 = \cdots = 0.$$

- So, to second order we can approximate

$$y(x) \simeq a_0 + a_2 x^2.$$

- We have fixed $y(1) = y_1$, so we can simplify to get

$$y(x) \simeq (y_1 - a_2) + a_2 x^2.$$

## Example 11.2 (Ritz and the Catenary-2)

$$
\begin{aligned}
y &\simeq y_1 - a_2 + a_2 x^2 \\
y' &\simeq 2 a_2 x
\end{aligned}
$$

- Taking into account $y(1) = 2$ we get $a_0 + a_2 = 2$. We can substitute into the functional

$$J_p[y] = mg \int\limits_{x_0}^{x_1} y\sqrt{1 + y'^2} dx$$

  and integrate to get a function $J_p[a_2]$ with respect to $a_2$.

- But this function is pretty complicated.

# Example 11.2 (Ritz and the Catenary-3)

From Maple we have the value for $J_p[a_2]$, $(a := a_2)$

> $f(x) := (2 - a + a\cdot x^2)\cdot\sqrt{1 + 4\cdot a^2\cdot x^2}$ :

> $int(f(x), x=-1..1)$

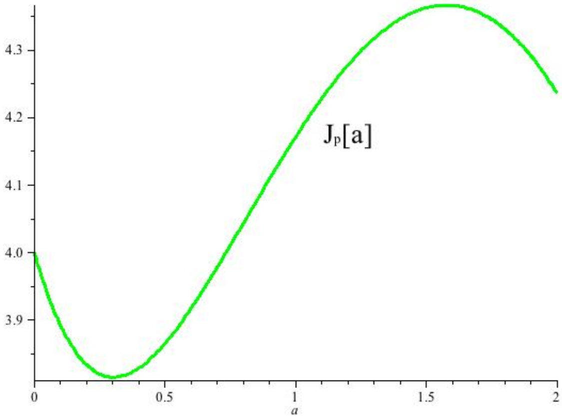
$$\frac{1}{64}\,\frac{1}{a^2}\Big(\Big(16\,a^2 \ln\big(\big(-2\,a + \sqrt{1+4\,a^2}\,\mathrm{csgn}(a)\big)\,\mathrm{csgn}(a)\big) + 128\sqrt{1+4\,a^2}\,a^2\,\mathrm{csgn}(a) - 64\,a^3\sqrt{1+4\,a^2}\,\mathrm{csgn}(a) - 32\,a\ln\big(\big(-2\,a + \sqrt{1+4\,a^2}\,\mathrm{csgn}(a)\big)\,\mathrm{csgn}(a)\big) + \ln\big(\big(-2\,a + \sqrt{1+4\,a^2}\,\mathrm{csgn}(a)\big)\,\mathrm{csgn}(a)\big) - 4\sqrt{1+4\,a^2}\,a\,\mathrm{csgn}(a) + 8\,(1+4\,a^2)^{3/2}\,a\,\mathrm{csgn}(a) - 16\,a^2\ln\big(\big(2\,a + \sqrt{1+4\,a^2}\,\mathrm{csgn}(a)\big)\,\mathrm{csgn}(a)\big) + 32\,a\ln\big(\big(2\,a + \sqrt{1+4\,a^2}\,\mathrm{csgn}(a)\big)\,\mathrm{csgn}(a)\big) - \ln\big(\big(2\,a + \sqrt{1+4\,a^2}\,\mathrm{csgn}(a)\big)\,\mathrm{csgn}(a)\big)\Big)\,\mathrm{csgn}(a)\Big) \tag{1}$$

>

### Example 11.2 (Ritz and the Catenary-4)

- Its a pain to find the zeros of $\frac{dJ_p}{da}$, but its easy to plot, and find them numerically.

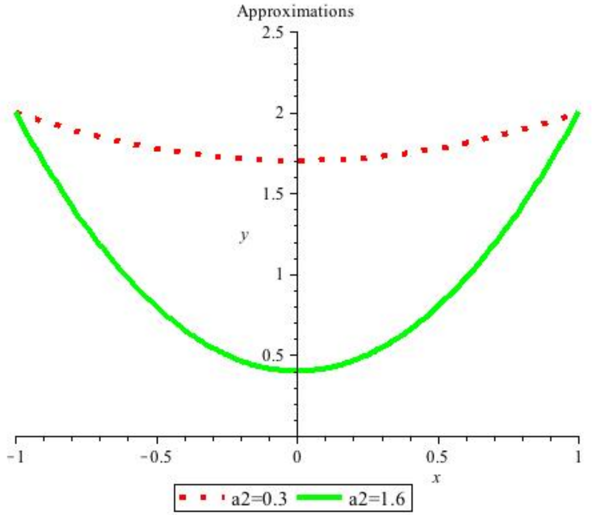
# Example 11.2 (Ritz and the Catenary-5)



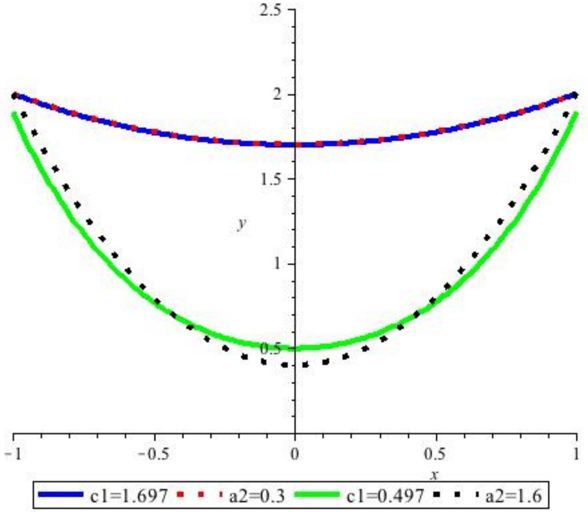

Stationary points

- local max: $a = a_2 \simeq 1.6$
- local min: $a = a_2 \simeq 0.3$

# Example 11.2 (Ritz and the Catenary-6)



## Example 11.2 (Ritz and the Catenary-7)



# Ritz and the Catenary

Doesn't just give us an approximation to the extremal curves, its also give us some insight into the nature of these extremals. If

- approximations are near to the actual extrema
- There are no other extrema so close by
- The functional is smooth (it can't have jumps either)

Then the type of extrema we get for the approximation will be the same for the real extrema, i.e.,

- local max: $a_2 \simeq 1.6 \Rightarrow$ local max for $c_1 = 0.497$
- local min: $a_2 \simeq 0.3 \Rightarrow$ local min for $c_1 = 1.697$

# More than one independent variables

## $2D$ Case:

We are approximating a surface with series of functions, e.g.

$$z(x,y) \simeq z_n(x,y) = \phi_0(x,y) + \sum_{i=1}^{n} c_i \phi_i(x,y)$$

where

- $\phi_0(x,y)$ satisfies the boundary conditions, e.g.

$$\phi_0(x,y) = z_0(x,y) \quad \text{for} \quad (x,y) \in \partial\Omega,$$

  the boundary of the region on interest $\Omega$,

- and the $\phi_i(x,y)$ satisfy the homogeneous boundary conditions

$$\phi_i(x,y) = 0 \quad \text{for} \quad (x,y) \in \partial\Omega.$$

## $2D$ Case:

- As before, we approximate the functional by

$$J[z] \simeq J[z_n] = J_n(c_1, \dots, c_n).$$

- As before we determine the $c_j$ by requiring that the partial derivatives are zero, e.g.

$$\frac{\partial J_n}{\partial c_i} = 0$$

for all $i = 1, 2, \dots, n$.

# Kantorovich's Method

- Approximate with

$$z(x, y) \simeq z_n(x, y) = \phi_0(x, y) + \sum_{i=1}^{n} c_i(x)\phi_i(x, y).$$

- Again the $\phi_i$ are suitable chosen, but the $c_i$ are no longer constants, but rather functions of one independent variable.
- This allows a larger class of functions to be used.

# Kantorovich's Method

- Note that the integral function

$$J[z_n] = \iint_{\Omega} z_n(x,y)dxdy = \sum_{i=0}^{n} \int c_i(x) \left[ \int_{y_0(x)}^{y_1(x)} \phi_i(x,y)dy \right] dx$$

- We integrate the inner integral, and get

$$J[z_n] = \sum_{i=0}^{n} \int c_i(x)\Phi_i(x)dx.$$

- Now we just have a function of $x$, and so we may apply the Euler-Lagrange machinery.
- The method approx. separates the variables $x$ and $y$.

## Example 11.3

Find the extremals of

$$J[z(x,y)] = \int_{-b}^{b}\int_{-a}^{a} \left(z_x^2 + z_y^2 - 2z\right) dxdy$$

with $z = 0$ on the boundary.

- The Euler-Lagrange equation reduces to the Poisson equation, e.g.

$$F_z - \frac{d}{dx}F_{z_x} - \frac{d}{dy}F_{z_y} = 0$$

$$-2 - \frac{d}{dx}(2z_x) - \frac{d}{dy}(2z_y) = 0$$

$$z_{xx} + z_{yy} = -1$$

## Example 11.3

- Approximate

$$z_1(x,y) = c(x)\left(b^2 - y^2\right)$$

- Note $z_1(x,\pm b) = 0$ (as required) and

$$\left(\frac{\partial z_1}{\partial x}\right)^2 = \left[c'(x)\left(b^2 - y^2\right)\right]^2 = c'(x)^2\left[b^4 - 2b^2y^2 + y^4\right],$$

$$\left(\frac{\partial z_1}{\partial y}\right)^2 = [c(x)2y]^2 = 4c(x)^2y^2$$

## Example 11.3

Hence, we approximate

$$J[z(x,y)] \simeq J[z_1(x,y)] = \int_{-b}^{b}\int_{-a}^{a}\left(z_x^2 + z_y^2 - 2z\right)dxdy$$

$$= \int_{-a}^{a}\left[\int_{-b}^{b}\left[c'(x)^2\left(b^2-y^2\right)^2 + 4c(x)^2y^2 - 2c(x)\left(b^2-y^2\right)\right]dy\right]dx$$

$$= \int_{-a}^{a}\Big[c'(x)^2\left(b^4y - 2b^2y^3/3 + y^5/5\right) + 4c(x)^2y^3/3$$

$$+2c(x)\left(b^2y - y^3/3\right)\Big]_{-b}^{b}dx$$

$$= \int_{-a}^{a}\left[\frac{16}{15}b^5c'(x)^2 + \frac{8}{3}b^3c(x)^2 - \frac{8}{3}b^3c(x)\right]dx$$

## Example 11.3

- So we can write

$$J[z(x,y)] \simeq J[z_1(x,y)] = J[c(x)] = \int_{-a}^{a} F(x,c,c')dx$$

- We can use the simple Euler-Lagrange equation, where

$$F(x,c,c') = \frac{16}{15}b^5 c'(x)^2 + \frac{8}{3}b^3 c(x)^2 - \frac{8}{3}b^3 c(x)$$

$$\frac{\partial F}{\partial c} = \frac{16}{3}b^3 c(x) - \frac{8}{3}b^3$$

$$\frac{\partial F}{\partial c'} = \frac{32}{15}b^5 c'(x) \qquad \frac{d}{dx}\frac{\partial F}{\partial c'} = \frac{32}{15}b^5 c''(x)$$

## Example 11.3

- The Euler-Lagrange equation

$$\frac{16}{3}b^3 c(x) - \frac{8}{3}b^3 - \frac{32}{15}b^5 c''(x) = 0$$

$$c''(x) - \frac{5}{2b^2}c(x) = -\frac{5}{4b^2}$$

- Solutions

$$c(x) = k_1 \cosh\left(\sqrt{\frac{5}{2}}\frac{x}{b}\right) + k_2 \sinh\left(\sqrt{\frac{5}{2}}\frac{x}{b}\right) + \frac{1}{2}$$

## Example 11.3

- Note that the function must be zero on the boundary, so $z(\pm a, y) = 0$.
- We look for an even function $c(x)$, and so $k_2 = 0$.
- Also $c(\pm a) = 0$, so

$$c(a) = k_1 \cosh\left(\sqrt{\frac{5}{2}}\frac{a}{b}\right) + \frac{1}{2}$$

$$-\frac{1}{2} = k_1 \cosh\left(\sqrt{\frac{5}{2}}\frac{a}{b}\right)$$

$$k_1 = -\frac{1}{2\cosh\left(\sqrt{\frac{5}{2}}\frac{a}{b}\right)}$$

## Example 11.3

- Solution

$$z_1(x,y) = \frac{1}{2}(b^2 - y^2)\left(1 - \frac{cosh\left(\sqrt{\frac{5}{2}}\frac{x}{b}\right)}{cosh\left(\sqrt{\frac{5}{2}}\frac{a}{b}\right)}\right)$$

- If we want a more exact approximation, we could try

$$z_2(x,y) = (b^2 - y^2)c_1(x) + (b^2 - y^2)^2 c_2(x).$$

## Remarks

- Obviously, quality of solution depends on
  - family of functions chosen
  - number of terms used, $n$
- Could test convergence by increasing $n$ and seeing the difference in
$$|J[y_{n+1}] - J[y_n]|,$$
but this is not guaranteed to be a good indication.
- A better way to assess convergence is to have a lower bound
$$\text{lower bound } \leqslant J[y] \leqslant \text{ upper bound}$$

# §12. Introduction in Optimal Control Problems

# Formulation of control problems

We break a control problems into two parts

1. The system state: $\mathbf{x}(t) = (x_1(t), x_2(t), \ldots, x_n(t))^t$

   The system state describes the system (e.g. position and velocity of the car in car parking example)

2. The control: $\mathbf{u}(t) = (u_1(t), \ldots, u_m(t))^t$

   We apply the control to the system (e.g. force applied to the car).

The evolution of the system is governed by the set of DEs

$$\dot{\mathbf{x}}(t) = \mathbf{g}(t, \mathbf{x}, \mathbf{u})$$

In a control problem we want to get the system to a particular state $\mathbf{x}(t)$ at time $t$, given initial state $\mathbf{x}(t_0)$.

# Optimal control problems

- In an optimal control problem we have still have the system equations
$$\dot{\mathbf{x}}(t) = \mathbf{g}(t, \mathbf{x}, \mathbf{u})$$
and we might wish to get to state $\mathbf{x}(t)$ given initial state $\mathbf{x}(t_0)$, but now we wish to do so while minimizing a functional
$$J[\mathbf{x}, \mathbf{u}] = \int_{t_0}^{t_1} F(t, \mathbf{x}, \mathbf{u})dt.$$
- That is, we wish to choose a function $\mathbf{u}(t)$ which minimizes the functional $J[\mathbf{x}, \mathbf{u}]$, while satisfying the end-point conditions $\mathbf{x}(t_0) = \mathbf{x}_0$ and $\mathbf{x}(t_1) = \mathbf{x}_1$, and the non-holonomic constaints
$$\dot{\mathbf{x}}(t) = \mathbf{g}(t, \mathbf{x}, \mathbf{u}).$$

# Optimal control problems

Optimization functional

$$J[\mathbf{x}, \mathbf{u}] = \int_{t_0}^{t_1} F(t, \mathbf{x}, \mathbf{u}) dt$$

## Remarks

Note that

- $F(t, \mathbf{x}, \mathbf{u})$ has no dependence on $\dot{\mathbf{u}}$: this is typically because costs depend on the control, not how we change the control, but there might be counter-examples.
- $F(t, \mathbf{x}, \mathbf{u})$ has no dependence on $\dot{\mathbf{x}}$: this is common in control problems, but not universal (we have seen at least one counter example).

# Terminal costs

- Sometimes in optimal control we don't fix the end-point $\mathbf{x}(t_1)$, but rather we assign a cost $\phi(t_1, \mathbf{x}(t_1))$ to particular end-points.
- So now we wish to choose a control $\mathbf{u}(t)$ which minimizes the functional

$$J[\mathbf{x}, \mathbf{u}] = \phi(t_1, \mathbf{x}(t_1)) + \int\limits_{t_0}^{t_1} F(t, \mathbf{x}, \mathbf{u}) dt$$

while satisfying the single end-point condition $x(t_0) = \mathbf{x}_0$, and the non-holonomic constraint $\dot{\mathbf{x}}(t) = \mathbf{g}(t, \mathbf{x}, \mathbf{u})$.

- $\phi(t_1, \mathbf{x}(t_1))$ is called the terminal cost.

# System Terminology

- **linear:** the state equations are a set of linear DEs.
- **autonomous:** time doesn't appear explicitly in the state equations (e.g. in $g(\mathbf{x}, \mathbf{u})$, or $F(\mathbf{x}, \mathbf{u})$).
  - also called time-invariant.
- **terminal cost:** the term $\phi(t_1, \mathbf{x}(t_1))$ is called the terminal cost.
- **controllable:** a solution to the control problem exists.
- **stable:** a stable equilibrium solution to the system DEs exists.
  - often we are interested in problems that are unstable, or we wouldn't really need a control.

# Control Terminology

- control (driver or automatic)
  - **planned** (open loop)
  - **feedback** (closed loop) control depends on current state
- type of control
  - movement from $A$ to $B$
  - continuous operations (maintain equilibrium)
- type of cost functional $J$
  - minimum time
  - minimum fuel
  - quadratic costs
- admissible controls
  - unbounded / bounded / bang-bang

# Cost functional examples

- **minimum time:** choose the fastest possible control

$$J[x, u] = \int_{t_0}^{t_1} dt.$$

- **minimum fuel:** fuel is expended by the controller, and we wish to minimize this

$$J[x, u] = \int_{t_0}^{t_1} |u(t)| dt$$

- **quadratic costs:**

$$J[x, u] = \int_{t_0}^{t_1} \left( x^2(t) + \alpha u^2(t) \right) dt$$

# Boundary conditions

- End time $t_1$: can be fixed or free
- End position $\mathbf{x}(t_1)$: can be fixed or free

In the cases with free boundary conditions, we introduce natural, or transversal boundary conditions.

## Example 12.1 Dynamic production

- A producer in purely competitive market
  - A large numbers of independent producers
  - Standardized product, e.g. potatoes
  - Firms are „price takers", i.e. they have no significant control over product price
  - Free entry and exit
  - Free flow of information
- wants to find optimal production path $x(t)$, $0 \leqslant t \leqslant T$.
- production target $x(T) = x_T$
- profit at time $t$ is $\pi(x, \dot{x}, t)$
- maximize profit functional $J[x] = \int\limits_0^T \pi(x, \dot{x}, t)dt$.

## Example 12.1 Dynamic production-2

Profit calculation

- quadratic production costs $C_1 = a_1 x^2 + b_1 x + c_1$
  - labor
  - raw materials
- production increase costs $C_2 = a_2 (\dot{x})^2 + b_2 \dot{x} + c_2$
  - new bildings
  - recruiting and training costs
- revenue $r = px$ where $p$ is the constant price per unit
  - $p = const$ due to purely competitive market
- profit at time $t$ is

$$\pi(x, \dot{x}, t) = px - C_1(x) - C_2(\dot{x}).$$

## Example 12.1 Dynamic production-3

Problem formulation: maximize total profit

$$J[x] = \int_0^T (px - C_1(x) - C_2(\dot{x}))\, dt$$

subject to $x(0) = 0$ and $x(T) = x_T$.

- notice that the control, and rate of change of state are the same (i.e., $u = \dot{x}$) but we write it as above for simplicity
- autonomous problem
- the control is planned, and has quadratic costs
- admissible controls are unbounded

## Example 12.1 Dynamic production-4

Euler-Lagrange equations

$$
\begin{aligned}
\frac{\partial \pi}{\partial x} - \frac{d}{dt}\frac{\partial \pi}{\partial \dot{x}} &= 0 \\
p - \frac{\partial C_1}{\partial x} + \frac{d}{dt}\frac{\partial C_2}{\partial \dot{x}} &= 0 \\
p - 2a_1 x - b_1 + \frac{d}{dt}\left[2a_2\dot{x} + b_2\right] &= 0 \\
2a_2\ddot{x} - 2a_1 x + p - b_1 &= 0 \\
\ddot{x} - \frac{a_1}{a_2}x &= \frac{b_1 - p}{2a_2}
\end{aligned}
$$

for $a_2 \neq 0$.

## Example 12.1 Dynamic production-5

Solution (for $a_1, a_2 \neq 0$)

$$x(t) = Ae^{\sqrt{\frac{a_1}{a_2}}t} + Be^{-\sqrt{\frac{a_1}{a_2}}t} + \frac{b_1 - p}{2a_2}$$

where $A$ and $B$ are determined by the fixed end points $x(0) = x_0$ and $x(T) = x_T$.

This gives the optimal production schedule

- no dependence on $c_1$ or $c_2$ (these are constant costs and so shouldn't effect production strategy)
- no dependence on $b_2$ because this is a linear cost in increasing production, and so occurs regardless of how we increase over time (to get to the final production target $x(T) = x_T$).

## Example 12.1 Dynamic production-6

What happens if we make the end point $x(T)$ free, i.e. we don't have a production target at time $T$?

Then we get a natural boundary condition

$$\left.\frac{\partial \pi}{\partial \dot{x}}\right|_{t=T} = \left.\frac{\partial C_2}{\partial \dot{x}}\right|_{t=T} = \left. 2a_2\dot{x} + b_2\right|_{t=T} = 0$$

So, rearranging, we get

$$\dot{x}(T) = -\frac{b_2}{2a_2}$$

- constants $A$ and $B$ are determined by end-point conditions $x(0) = 0$ and $\dot{x}(T) = -\frac{b_2}{2a_2}$.

•Production costs

$$C_1 = x^2 + 5x$$

•Production increase costs

$$C_2 = 2\dot{x}^2 + 5\dot{x}$$

- $p = 10$
- $T = 1$
- $x_0 = 0, \quad x_T = 1$