

Calculus of Variations

Summer Term 2017

Lecture 15

Universität des Saarlandes

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Purpose of Lesson:

- To consider aerospace example (cont.)
- Hamiltonian's formulation

Natural boundary conditions

Example 12.2 Launching a rocket-17

Given $\lambda_v(T) = 0$, and from previous work

$$\lambda_v = -\frac{\lambda_z v}{g} - \lambda_z t + b$$

we get

$$\begin{aligned} \frac{\lambda_z v(T)}{g} &= -\lambda_z T + b \\ &= \lambda_u \tan \theta(T) \end{aligned}$$

$$v(T) = \frac{\lambda_u g}{\lambda_z} \tan \theta(T)$$

Natural boundary conditions

Example 12.2 Launching a rocket-18

$$H(T) = [F - \dot{u}\lambda_u - \dot{v}\lambda_v - \dot{z}\lambda_z]_{t=T} = 0.$$

Substituting F and taking into account that $\lambda_v(T) = 0$ we get

$$a\lambda_u \cos \theta(T) + a \frac{\lambda_z v(T)}{g} \sin \theta(T) = 1$$

Combining the latter with $v(T) = \frac{\lambda_u g}{\lambda_z} \tan \theta(T)$ we arrive at

$$a\lambda_u \cos \theta(T) + a\lambda_u \tan \theta(T) \sin \theta(T) = 1$$

$$\lambda_u = \frac{\cos \theta(T)}{a}$$

Acceleration profile

Example 12.2 Launching a rocket-19

The next step depend on the acceleration profile $a(t)$, but lets take a simple case $a = \text{const}$.

First we can solve the DEs, with respect to θ , using the chain rule

$$\frac{dX}{dt} = \frac{dX}{d\theta} \frac{d\theta}{dt} = -\cos^2 \theta \frac{\lambda_z}{\lambda_u} \frac{dX}{d\theta}$$

e.g. from the system DE $\dot{u} = a \cos \theta$

$$\dot{u} = -\cos^2 \theta \frac{\lambda_z}{\lambda_u} \frac{du}{d\theta}$$

$$\frac{du}{d\theta} = -\frac{\lambda_u}{\lambda_z \cos^2 \theta} \dot{u} = -\frac{a \lambda_u}{\lambda_z \cos \theta}$$

Acceleration profile

Example 12.2 Launching a rocket-20

$$\frac{dX}{d\theta} = \frac{\frac{dX}{dt}}{\frac{d\theta}{dt}} = \frac{\frac{dX}{dt}}{-\cos^2 \theta \frac{\lambda_z}{\lambda_u}}$$

The complete set of system DEs becomes

$$\begin{aligned}\frac{du}{d\theta} &= -\frac{a\lambda_u}{\lambda_z \cos \theta} \\ \frac{dv}{d\theta} &= -\frac{a\lambda_u}{\lambda_z} \frac{\sin \theta}{\cos^2 \theta} + \frac{g\lambda_u}{\lambda_z \cos^2 \theta} \\ \frac{dz}{d\theta} &= -\frac{a\lambda_u}{g\lambda_z} \frac{\sin \theta}{\cos^2 \theta} v(\theta)\end{aligned}$$

These can just be integrated with respect to θ .

Acceleration profile

Example 12.2 Launching a rocket-21

The system DEs can be directly integrated (with respect to θ) including initial conditions $u(0) = v(0) = z(0) = 0$ to get

$$u(\theta) = \frac{a\lambda_u}{\lambda_z} \log \left(\frac{\sec \theta_0 + \tan \theta_0}{\sec \theta + \tan \theta} \right)$$

$$v(\theta) = \frac{a\lambda_u}{\lambda_z} (\sec \theta_0 - \sec \theta) - \frac{g\lambda_u}{\lambda_z} (\tan \theta_0 - \tan \theta)$$

$$z(\theta) = \frac{a^2\lambda_u^2}{g\lambda_z^2} \sec \theta_0 (\sec \theta_0 - \sec \theta) - \frac{a^2\lambda_u^2}{2g\lambda_z^2} (\tan^2 \theta_0 - \tan^2 \theta) \\ + \frac{a\lambda_u^2}{2\lambda_z^2} \left[\tan \theta_0 \sec \theta_0 - \tan \theta \sec \theta + \log \left(\frac{\sec \theta_0 + \tan \theta_0}{\sec \theta + \tan \theta} \right) \right]$$

$$\theta = \tan^{-1} \left(-\frac{\lambda_z t - b}{\lambda_u} \right)$$

Calculating the constants

Example 12.2 Launching a rocket-22

There are **five** constants to calculate:

- θ_0 the initial angle of thrust
- θ_1 the final angle of thrust
- λ_u
- λ_z
- b

and also we need to calculate T .

Solving for end-point conditions is non-trivial, but a method that works well follows.

Calculating the constants

Example 12.2 Launching a rocket-23

Take the equation for v at time T , and substitute $\lambda_z v(T) = g\lambda_u \tan \theta_1$ to get

$$v(\theta_1) = \frac{a\lambda_u}{\lambda_z} (\sec \theta_0 - \sec \theta_1) - \frac{g\lambda_u}{\lambda_z} (\tan \theta_0 - \tan \theta_1)$$

$$\frac{g\lambda_u}{\lambda_z} \tan \theta_1 = \frac{a\lambda_u}{\lambda_z} (\sec \theta_0 - \sec \theta_1) - \frac{g\lambda_u}{\lambda_z} (\tan \theta_0 - \tan \theta_1)$$

$$\sec \theta_1 = \sec \theta_0 - \frac{g}{a} \tan \theta_0$$

which gives us a way to calculate θ_1 from θ_0 .

Once we know θ_1 we can calculate λ_u using $a\lambda_u = \cos \theta_1$, and b from $\tan \theta = -(\lambda_z t - b)/\lambda_u$ at $t = 0$.

Then we can calculate λ_z from $u(\theta_1) = u_0$, the orbital injection velocity.

Calculating the constants

Example 12.2 Launching a rocket-24

So the only remaining question is how to calculate θ_0 . We do so numerically, by

- take a range of θ_0
- calculate all of the above
- use this to calculate $z(T) = z_1$ as a function of θ_0
- look for the point where $z_1(\theta_0) = h$ the orbit height.

That gives us the θ_0 , from which we can derive everything else.

Restricting choice of θ_0

Example 12.2 Launching a rocket-25

Calculating the range of θ_0 to search

- The maximum (reasonable) value for θ_0 is $\pi/2$.
- The minimum value of θ_0 will be determined by the minimum possible value of θ_1 , i.e., $\theta_1 = 0$

$$\sec \theta_1 = \sec \theta_0 - \frac{g}{a} \tan \theta_0$$

$$\sec 0 = 1 = \sec \theta_0 - \frac{g}{a} \tan \theta_0$$

$$1 = \frac{1 + \tan^2(\theta_0/2)}{1 - \tan^2(\theta_0/2)} - \frac{g}{a} \frac{2 \tan(\theta_0/2)}{1 - \tan^2(\theta_0/2)}$$

$$1 - \tan^2(\theta_0/2) = 1 + \tan^2(\theta_0/2) - \frac{2g}{a} \tan(\theta_0/2)$$

Restricting choice of θ_0

Example 12.2 Launching a rocket-26

$$1 - \tan^2 (\theta_0/2) = 1 + \tan^2 (\theta_0/2) - \frac{2g}{a} \tan (\theta_0/2)$$

$$2 \tan^2 (\theta_0/2) - \frac{2g}{a} \tan (\theta_0/2) = 0$$

$$\tan (\theta_0/2) \left(\tan (\theta_0/2) - \frac{g}{a} \right) = 0.$$

Now θ_0 can't be zero, so the last step implies that the minimum value of θ_0 is

$$\theta_0 = 2 \tan^{-1} \left(\frac{g}{a} \right).$$

Note the existence of a minimum critical h below which we can't find a trajectory of this type.

Parameters

Parameters of previous example consistent with a LEO.

$$h = 500km$$

$$u_0 = 8000m/s$$

$$g = 9.8m/s^2$$

$$a = 3g$$

Derived constants

$$\theta_0 = 0.234\pi$$

$$\theta_1 = 0.0973\pi$$

$$\lambda_u = 0.0324$$

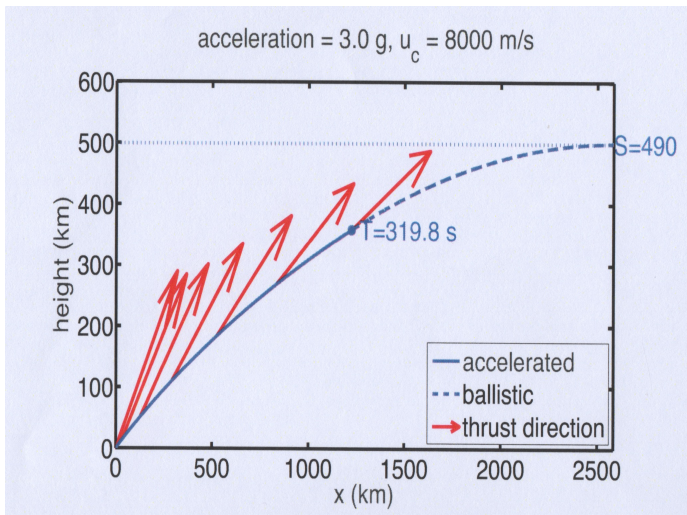
$$\lambda_z = 6.0257e - 0.5$$

$$b = -0.0295$$

$$T = 319.8 \text{ seconds}$$

$$S = 489.6 \text{ seconds}$$

Trajectory



Generalizations

More realistic assumptions

- non-zero drag (depends on velocity and height)
- Thrust is constant, but rocket mass changes, so that acceleration isn't constant
- multiple stages
- centripetal forces

§13. Hamiltonian's formulation and Pontryagin's Maximum Principle

Hamiltonian's formulation

- We've seen the Hamiltonian \mathcal{H} earlier on, but haven't explored its full power.
- Using \mathcal{H} can often result in a simpler approach than solving the E-L equations, e.g., where F has no dependence on x , or where there is more than one dependent variable.
- Hamiltonian's formulation can lead to an understanding of how symmetries in the problem of interest lead to conservation laws.

Legendre transformation

- transformation that depends on the derivatives of a variable
- simple one variable Legendre transform of

$$y : [x_0, x_1] \rightarrow \mathbb{R},$$

by defining new variable p , by

$$p(x) = y'(x)$$

- provided $y''(x) > 0$ we can define x in terms of p , by introducing the Hamiltonian

$$\mathbb{H}(p) = px - y(x)$$

Legendre transformation

Assume for convenience that y is convex, e.g. $y'' > 0$ for $x \in [x_0, x_1]$.
Then

$$\begin{aligned}
 \frac{d\mathbb{H}}{dp} &= \frac{d}{dp}(xp) - \frac{dy}{dp} \\
 &= p \frac{dx}{dp} + x - \frac{dy}{dp} \\
 &= p \frac{dx}{dp} + x - \frac{dy}{dx} \frac{dx}{dp} \\
 &= \left(p - \frac{dy}{dx} \right) \frac{dx}{dp} + x \\
 &= x
 \end{aligned}$$

and also note $px - \mathbb{H} = y$, so from the pair (p, \mathbb{H}) we can recover the original pair (x, y) , by a Legendre transform.

Hamiltonian's formulation

Refer back to problems with more than one dependent variable, or where F has no dependence on x .

Define **generalized coordinates** $\mathbf{q} : [t_0, t_1] \rightarrow \mathbb{R}^n$.

- i.e. take a set of n functions $q_k(t)$, with two continuous derivatives with respect to t , and put them into a vector $\mathbf{q}(t)$
- dot notation

$$\dot{q}_k = \frac{dq_k}{dt}, \quad \ddot{q}_k = \frac{d^2q_k}{dt^2} \quad \text{and} \quad \dot{\mathbf{q}} = \left(\frac{dq_1}{dt}, \frac{dq_2}{dt}, \dots, \frac{dq_n}{dt} \right)$$

- Lagrangian $L(t, \mathbf{q}, \dot{\mathbf{q}})$

Hamilton's formulation

The extremal of the functional

$$J[\mathbf{q}] = \int_{t_0}^{t_1} L(t, \mathbf{q}, \dot{\mathbf{q}}) dt$$

satisfy the Euler-Lagrange equations

$$\frac{\partial L}{\partial q_k} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} = 0$$

for all k .

Hamilton's formulation

Legendre transform introduces the **conjugate** variables

$$p_i = \frac{\partial L}{\partial \dot{q}_i}.$$

Suppose these equations can be solved to write \dot{q}_i as a function of (t, q_i, p_i) , then the **Hamiltonian** is

$$\mathbb{H}(t, q_1, \dots, q_n, p_1, \dots, p_n) = \sum_{i=1}^n p_i \dot{q}_i - L(t, \mathbf{q}, \dot{\mathbf{q}}).$$

- the p_i are called **generalized momenta**

Hamilton's formulation

$$\mathbb{H}(t, q_1, \dots, q_n, p_1, \dots, p_n) = \sum_{i=1}^n p_i \dot{q}_i - L(t, \mathbf{q}, \dot{\mathbf{q}}).$$

So

$$\begin{aligned} \frac{\partial \mathbb{H}}{\partial p_i} &= \dot{q}_i \\ \frac{\partial \mathbb{H}}{\partial q_i} &= -\frac{\partial L}{\partial q_i} \end{aligned}$$

Given the E-L equations, the second equation gives

$$\frac{\partial \mathbb{H}}{\partial q_i} = -\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = -\frac{dp_i}{dt}.$$

Canonical Euler-Lagrange equations

$$\frac{\partial \mathbb{H}}{\partial p_i} = \frac{dq_i}{dt}$$
$$\frac{\partial \mathbb{H}}{\partial q_i} = -\frac{dp_i}{dt}$$

- called **Hamiltonian's equations** or **canonical Euler-Lagrange equations**.
- The n E-L DEs converted into $2n$ first-order DEs
- derivatives are now uncoupled
 - therefore may be easier to solve

Canonical Euler-Lagrange equations

We can get the same canonical E-L equations from finding extremals of the functional of $2n$ variables

$$\widehat{J}[q_1, \dots, q_n, p_1, \dots, p_n] = \int_a^b \left[\sum_{i=1}^n p_i \dot{q}_i - \mathbb{H} \right] dx$$

E.G.

$$\left(\frac{\partial}{\partial q_i} - \frac{d}{dt} \frac{\partial}{\partial \dot{q}_i} \right) \left[\sum_{i=1}^n p_i \dot{q}_i - \mathbb{H} \right] = 0$$

$$\left(\frac{\partial}{\partial p_i} - \frac{d}{dt} \frac{\partial}{\partial \dot{p}_i} \right) \left[\sum_{i=1}^n p_i \dot{q}_i - \mathbb{H} \right] = 0$$

Hamilton's formulation

- J and \hat{J} are equivalent under the Legendre transformation
 - make q and p independent, whereas before it was a bit of trick to pretend q and \dot{q}_i were independent
- If L does not depend on t , then it should be clear from the Legendre transformation that \mathbb{H} won't depend on t
 - the system will be **conservative**
 - i.e. \mathbb{H} is a conserved (constant) quantity