

# Calculus of Variations

## Summer Term 2017

### Lecture 18

Universität des Saarlandes

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## Purpose of Lesson:

- To introduce direct methods
- To discuss strong and weak topologies in real Banach spaces.

# Direct Methods:

- The relevant ideas developed during the 20th century are called **Direct Methods**.
- An important ingredient here is the introduction of functional analytic techniques.
- In fact, it was the Calculus of Variations, which gave birth to the theory of Functional Analysis.

# The Main Idea of Direct Method:

Consider a minimization problem on some class  $\mathcal{A}$  of functions:

$$\min_{u \in \mathcal{A}} J[u]$$

- Suppose there exist a minimizing sequence  $\{u_n\}$  in  $\mathcal{A}$ , i.e.

$$\lim_{n \rightarrow \infty} J[u_n] = \min_{u \in \mathcal{A}} J[u] < +\infty.$$

- Suppose we could find an element  $u_0$  in  $\mathcal{A}$  such that

$$"u_n \rightarrow u_0" \text{ as } n \rightarrow \infty$$

- Suppose the functional  $J$  has **some kind of continuity**. Therefore

$$\lim_{n \rightarrow \infty} J[u_n] = J[u_0].$$

- We conclude that  $u_0$  is a minimizer of the functional  $J$  because

$$J[u_0] = \lim_{n \rightarrow \infty} J[u_n] = \min_{u \in \mathcal{A}} J[u].$$

### Example 14.2 ( $J$ is not bounded below)

Let  $[0, \pi] \subset \mathbb{R}$  and  $\mathcal{A} = \{u \in C^1([0, \pi]) : u(0) = u(\pi) = 0\}$ .

Take

$$J[u] = \int_0^{\pi} [u'^2 - 2u^2] dx \quad \text{and} \quad u_n(x) = n \sin x \in \mathcal{A}.$$

But  $J[u_n] = -\frac{\pi n^2}{2} \rightarrow -\infty$  as  $n \rightarrow \infty$ . (Also,  $u_n$  is unbounded.)

### Example 14.3 (No convergent subsequence in minimizing sequence)

Let  $[0, \pi] \subset \mathbb{R}$  and  $\mathcal{A} = \{u \in C^1([0, \pi]) : u(0) = u(\pi) = 0\}$ .

Take

$$J[u] = \int_0^{\pi} (u'^2 - 1)^2 dx,$$

$$u_n(x) = \sqrt{\frac{1}{n^2} + \frac{\pi^2}{4}} - \sqrt{\frac{1}{n^2} + \left(x - \frac{\pi}{2}\right)^2} \in \mathcal{A},$$

$$J[u_n] \rightarrow 0 = \inf_{v \in \mathcal{A}} J[v].$$

But every subsequence converges to  $\frac{\pi}{2} - |x - \frac{\pi}{2}|$  which is not in  $\mathcal{A}$ .

### Example 14.4 ( $J$ is not lower semi-continuous)

Let  $[0, \pi] \subset \mathbb{R}$  and  $\mathcal{A} = \{u \in C^1([0, \pi]) : u(0) = u(\pi) = 0\}$ .

Let

$$g(p) = \begin{cases} p^2, & \text{if } p \neq 0; \\ 1, & \text{if } p = 0. \end{cases}$$

Take

$$J[u] = \int_0^{\pi} g(u') dx \quad \text{and} \quad u_n(x) = \frac{1}{n} \sin x \rightarrow 0.$$

Then  $u_n(x) \in \mathcal{A}$  and  $J[u_n] = \frac{\pi}{2n^2} \rightarrow 0 = l$  as  $n \rightarrow \infty$  but

$$\pi = J[0] = J[\lim_{n' \rightarrow \infty} u_{n'}] > \lim_{n \rightarrow \infty} J[u_n] = 0.$$

## §15. Suitable Function Spaces



# Topologies on Banach Spaces

- Let  $(\mathbb{X}, |\cdot|)$  denote a real Banach space.
- A Banach space is a complete, normed linear space. Complete means that any Cauchy sequence is convergent.
- A sequence  $\{x_n\}$  of real numbers is called a Cauchy sequence, if for every positive real number  $\varepsilon$ , there is a positive integer  $N$  such that for all natural numbers  $m, n > N$

$$|x_n - x_m| < \varepsilon.$$

- Examples:

1  $\mathbb{R}^n$  with the norm defined by  $\|x\| = \left(\sum_i |x_i|^2\right)^{1/2}$ .

- 2  $C^k(\overline{\Omega})$  with the norm defined by

$$\|u\|_{C^k} = \max_{0 \leq |\alpha| \leq k} \sup_{x \in \overline{\Omega}} |D^\alpha u(x)|$$

- We denote by  $\mathbb{X}'$  the topological dual space of  $\mathbb{X}$ :

$$\mathbb{X}' = \left\{ l : \mathbb{X} \rightarrow \mathbb{R} \text{ linear such that } \|l\|_{\mathbb{X}'} = \sup_{x \neq 0} \frac{|l(x)|}{|x|_{\mathbb{X}}} < \infty \right\}.$$

- Classically,  $\mathbb{X}$  can be endowed with two topologies.

### Definition (topologies on $\mathbb{X}$ )

- (i) The **strong topology**, denoted by  $x_n \xrightarrow{\mathbb{X}} x$ , is defined by

$$|x_n - x|_{\mathbb{X}} \rightarrow 0 \quad (n \rightarrow \infty).$$

- (ii) The **weak topology**, denoted by  $x_n \xrightarrow{\mathbb{X}} x$ , is defined by

$$l(x_n) \rightarrow l(x) \quad (n \rightarrow \infty) \quad \text{for every } l \in \mathbb{X}'.$$

- Strong convergence implies weak convergence, but the converse is false in general.

### Example 15.1 (Counterexample)

- Consider the sequence  $f_n(x) = \sin(2\pi xn)$ ,  $x \in (0, 1)$ , as  $n \rightarrow \infty$ .

- $f_n \xrightarrow{L_2(\Omega)} 0$ . To prove weak convergence, we have for all  $\varphi \in C^1(0, 1)$ , thanks to a classical integration by parts,

$$\int_0^1 \sin(2\pi xn) \varphi(x) dx = \frac{1}{2\pi n} [\varphi(0) - \varphi(1)] + \frac{1}{2\pi n} \int_0^1 \cos(2\pi xn) \varphi'(x) dx$$

## Example 15.2 (continued)

- So, it is clear that  $\langle f_n, \varphi \rangle \rightarrow 0$  as  $n \rightarrow \infty$  for all  $\varphi \in C^1(0, 1)$ , where  $\langle \cdot, \cdot \rangle$  is the usual scalar product in  $L_2(0, 1)$ .
- By density, this result can be generalized for all  $\varphi \in L_2(0, 1)$ .
- There is no strong convergence, i.e.,  $f_n(x) \xrightarrow{L_2(0,1)} 0$  is **not true**.
- To prove that there is no strong convergence, we observe that

$$\int_0^1 \sin^2(2\pi xn) dx = \frac{1}{2} \int_0^1 (1 - \cos[(4\pi xn)]) dx = \frac{1}{2}.$$

- The dual space  $\mathbb{X}'$  can also be endowed with the strong and the weak topologies.

### Definition (topologies on $\mathbb{X}'$ )

- (i) The strong topology, denoted by  $l_n \xrightarrow{\mathbb{X}'} l$ , is defined by

$$|l_n - l|_{\mathbb{X}'} \rightarrow 0, \text{ or equivalently, } \sup_{x \neq 0} \frac{|l_n(x) - l(x)|}{|x|_{\mathbb{X}}} \rightarrow 0 \quad (n \rightarrow \infty).$$

- (ii) The weak topology, denoted by  $l_n \xrightarrow{\mathbb{X}'} l$ , is defined by

$$z(l_n) \rightarrow z(l) \quad (n \rightarrow \infty) \quad \text{for every } z \in (\mathbb{X}')',$$

where  $(\mathbb{X}')'$  denotes the bidual space of  $\mathbb{X}$ .

- In some cases it is more convenient to equip  $\mathbb{X}'$  with a third topology:

The weak\* topology, denoted by  $l_n \xrightarrow[\mathbb{X}'^*]{} l$ , is defined by

$$l_n(x) \rightarrow l(x) \quad (n \rightarrow \infty) \quad \text{for every } x \in \mathbb{X}.$$

- The space  $\mathbb{X}$  is called **reflexive** if  $(\mathbb{X}')' = \mathbb{X}$ .
- $\mathbb{X}$  is called **separable** if it contains a countable dense set.
- Examples:
  - 1  $\mathbb{X} = L_p(\Omega)$  is reflexive for  $1 < p < \infty$  and separable for  $1 \leq p < \infty$ .  
The dual space of  $L_p(\Omega)$  is  $L_{p'}(\Omega)$  for  $1 \leq p < \infty$  with  $\frac{1}{p} + \frac{1}{p'} = 1$ .
  - 2  $\mathbb{X} = L_1(\Omega)$  is nonreflexive and  $\mathbb{X}' = L_\infty(\Omega)$ .

The main properties associated with these different topologies are summarized in the following theorem :

### Theorem 15.1 (weak sequential compactness)

- (i) Let  $\mathbb{X}$  be a reflexive Banach space,  $K > 0$ , and  $x_n \in \mathbb{X}$  a sequence such that  $|x_n|_{\mathbb{X}} \leq K$ .

Then there exist  $x \in \mathbb{X}$  and a subsequence  $x_{n_j}$  of  $x_n$  such that

$$x_{n_j} \xrightarrow{\mathbb{X}} x \quad (n \rightarrow \infty).$$

- (ii) Let  $\mathbb{X}$  be a separable Banach space,  $K > 0$ , and  $l_n \in \mathbb{X}'$  such that  $|l_n|_{\mathbb{X}'} \leq K$ .

Then there exist  $l \in \mathbb{X}'$  and a subsequence  $l_{n_j}$  of  $l_n$  such that

$$l_{n_j} \xrightarrow{\mathbb{X}'^*} l \quad (n \rightarrow \infty).$$