

# Calculus of Variations

## Summer Term 2017

### Lecture 2

Universität des Saarlandes

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## Purpose of Lesson:

- To discuss the generalizations of the E-L equations to case of  $n$  functions and to the ones of higher order derivatives.
- To discuss the special cases of the E-L equation.

## §2. Remarks on the Euler-Lagrange equation

# The Euler–Lagrange Equation:

If  $y$  minimizes  $J[y] = \int_a^b F(x, y, y') dx$ , then  $y$  **must** satisfy the equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx} \left[ \frac{\partial F}{\partial y'} \right] = 0$$

# History of Leonhard Euler and Joseph-Louis Lagrange



Leonhard Euler (1707-1783)



Joseph-Louis Lagrange (1736-1813)

- **Euler** developed Euler's Equations for fluids flow and Euler's formula

$$e^{ix} = \cos x + i \sin x.$$

In 1744 Euler published the first book on Calculus of Variations.



- **Lagrange** developed Lagrange Multipliers, Lagrangian Mechanics, and the Method of Variations of Parameters.

- In 1766 Lagrange succeeded Euler as the director of Mathematics at the Prussian Academy of Sciences in Berlin.
- In letters to Euler between 1754 and 1756 Lagrange shared his observation of a connection between minimizing functionals and finding extrema of a function.
- Euler was so impressed with Lagrange's simplification of his earlier analysis it is rumored he refrained from submitting a paper covering the same topics to give Lagrange more time.

## Generalization to $n$ functions:

- Let  $F(x, y_1, \dots, y_n, y_1', \dots, y_n')$  be a function with continuous first and second partial derivatives. Consider the problem of finding **necessary conditions** for finding the extremum of the following functional

$$J[y_1, \dots, y_n] = \int_a^b F(x, y_1, \dots, y_n, y_1', \dots, y_n') dx \rightarrow \min$$

which depends continuously on  $n$  continuously differentiable functions  $y_1(x), \dots, y_n(x)$  satisfying boundary conditions

$$y_i(a) = A_i, \quad y_i(b) = B_i \quad \text{for } i = 1, 2, \dots, n.$$

- One can derive that the necessary condition is a system of **Euler-Lagrange Equations**

$$\frac{\partial F}{\partial y_i} - \frac{d}{dx} \frac{\partial F}{\partial y_i'} = 0$$



## Generalization to Higher Order Derivatives:

- Let  $F(x, y, y', y'', \dots, y^{(n)})$  be a function with continuous first and second derivatives with respect to all arguments, and consider a functional of the form

$$J[y] = \int_a^b F(x, y, y', y'', \dots, y^{(n)}) dx \rightarrow \min$$

where the admissible class will be

$$\mathbb{A} = \{y(x) \in C^n[a, b]\}$$

$$y(a) = A_0, y'(a) = A_1, \dots, y^{(n-1)}(a) = A_{n-1}$$

$$y(b) = B_0, y'(b) = B_1, \dots, y^{(n-1)}(b) = B_{n-1}$$

- The necessary condition is the **Euler-Lagrange Equation**

$$F_y - \frac{d}{dx} F_{y'} + \frac{d^2}{dx^2} F_{y''} - \dots + (-1)^n \frac{d^n}{dx^n} F_{y^{(n)}} = 0$$

## Remark

For a functional of the form

$$J[y] = \int_a^b F(x, y, y') dx$$

the Euler-Lagrange equation is in general a second-order differential equation, but it may turn out that the curve for which the functional has its extremum is **not twice differentiable**.

## Example 2.1

Consider the functional

$$J[y] = \int_{-1}^1 y^2(2x - y')^2 dx$$

where

$$y(-1) = 0, \quad y(1) = 1.$$

- The minimum of  $J[y]$  equals zero and is achieved for the function

$$y = y(x) = \begin{cases} 0 & \text{for } -1 \leq x \leq 0, \\ x^2 & \text{for } 0 < x \leq 1, \end{cases}$$

which has no second derivative for  $x = 0$ .

- Nevertheless,  $y(x)$  satisfies the appropriate E-L equation.
- In fact, since in this case

$$F(x, y, y') = y^2(2x - y')^2$$

it follows that all the functions

$$F_y = 2y(2x - y')^2, \quad F_{y'} = -2y^2(2x - y'), \quad \frac{d}{dx}F_{y'}$$

vanish identically for  $-1 \leq x \leq 1$ .

- Thus, despite of the fact that the E-L equation is of the second order and  $y''(x)$  does not exist everywhere in  $[-1, 1]$ , substitution of  $y(x)$  into E-L's equation converts it into an identity.

We now give conditions guaranteeing that a solution of the E-L equation has a second derivative:

### Theorem 2.1

Suppose  $y = y(x)$  has a continuous first derivative and satisfy the E-L equation

$$F_y - \frac{d}{dx} F_{y'} = 0.$$

Then, if the function  $F(x, y, y')$  has continuous first and second derivatives with respect to all its arguments,  $y(x)$  has a continuous second derivative at all points  $(x, y)$  where

$$F_{y'y'}(x, y(x), y'(x)) \neq 0.$$

# Special Cases of the E-L Equation

We now indicate some special cases where the Euler-Lagrange equation can be reduced to a first-order differential equation, or where its solution can be obtained by evaluating integrals.

1. Suppose the integrand does not depend on  $y$ , i.e., let the functional under consideration have the form

$$J[y] = \int_a^b F(x, y') dx,$$

where  $F$  does not contain  $y$  explicitly.

- In this case, the E-L equation becomes  $\frac{d}{dx} F_{y'} = 0$ , which obviously has the first integral

$$F_{y'} = C, \quad (2.1)$$

where  $C$  is a constant. This is a first-order ODE which does not contain  $y$ . Solving (2.1) for  $y'$ , we obtain an equation of the form

$$y' = f(x, C).$$

2. If the integrand does not depend on  $x$ , i.e., if

$$J[y] = \int_a^b F(y, y') dx,$$

then

$$F_y - \frac{d}{dx} F_{y'} = F_y - F_{y'y} y' - F_{y'y'} y''. \quad (2.2)$$

Multiplying (2.2) by  $y'$ , we obtain

$$F_y y' - F_{y'y} y'^2 - F_{y'y'} y' y'' = \frac{d}{dx} (F - y' F_{y'}).$$

Thus, in this case the E-L equation has the first integral

$$F - y' F_{y'} = C$$

where  $C$  is a constant.