# Calculus of Variations
# Summer Term 2017

Lecture 20

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## Purpose of Lesson:

- To discuss several examples emphasizing the optimality of the hypothesis of the existence theorem.
- To introduce the relaxation theory and present a way of defining a "generalized" solution of our standard minimization problem.

We propose below some classical examples where either coercivity, reflexivity, or convexity is no longer true.

### Example 16.1 (Weierstrass)

- Let $\Omega = (0, 1]$, let $f$ be defined by $f(x, u, \xi) = x\xi^2$ and let us set

$$J[u] = \int_0^1 x u'^2 dx \qquad \text{with} \quad u(0) = 1, \quad u(1) = 0.$$

- Then, we can show that this problem does not have any solution.

## Example 16.1 (continued)

- The function $f$ is convex, but the $W_2^1(\Omega)$-coercivity with respect to $u$ is not satisfied because the integrand $f(x,\xi) = x\xi^2$ vanishes at $x = 0$.

- Let us first prove that

$$m := \inf J[u] = 0.$$

The idea is to propose the following minimizing seqience

$$u_n(x) = \begin{cases} 1, & \text{if} \quad x \in (0, 1/n), \\ -\dfrac{\log(x)}{\log(n)}, & \text{if} \quad x \in (1/n, 1). \end{cases}$$

## Example 16.1 (continued)

- It is then easy to verify that $u_n \in W^1_\infty(0,1)$, and that

$$J[u_n] = \int\limits_0^1 x u_n'^2 dx = \frac{1}{\log(n)} \to 0.$$

So we have $m = 0$.

- If there exists a minimum $u_0$, then we should have $J[u_0] = 0$, that is $u_0' = 0$ almost everywhere (a.e.) in $(0,1)$.
- But $u_0' = 0$ a.e. in $(0,1)$ is clearly incompatible with the boundary conditions.

### Example 16.2 (Bolza)

- Let $\Omega = (0,1]$, and let $f$ be defined by $f(x,u,\xi) = u^2 + (\xi^2 - 1)^2$. The Bolza problem is

$$\min \leftarrow J[u] = \int_0^1 \left[ (1 - u'^2)^2 + u^2 \right] dx, \quad u \in W_4^1(0,1)$$

$$\text{with} \quad u(0) = u(1) = 0.$$

- The functional $J$ is clearly nonconvex.

## Example 16.2 (continued)

- It is easy to see that

$$m := \inf J[u] = 0.$$

Indeed, for $n$ an integer and $0 \leqslant k \leqslant n-1$, if we choose

$$u_n(x) = \begin{cases} x - \dfrac{k}{n}, & \text{if} \quad x \in \left(\dfrac{2k}{2n}, \dfrac{2k+1}{2n}\right) \\ -x + \dfrac{k+1}{n}, & \text{if} \quad x \in \left(\dfrac{2k+1}{2n}, \dfrac{2k+2}{2n}\right) \end{cases},$$

then $u_n \in W^1_\infty(0,1)$ and

$$0 \leqslant u_n(x) \leqslant \frac{1}{2n} \quad \text{for every } x \in (0,1),$$
$$|u_n'(x)| = 1 \quad \text{a.e. in } (0,1),$$
$$u_n(0) = u_n(1) = 0.$$

### Example 16.2 (continued)

- Therefore,
$$0 \leqslant \inf_{u} J[u] \leqslant J[u_n] \leqslant \frac{1}{4n^2}.$$
Letting $n \to \infty$, we obtain $m = 0$.

- However, there exists no function $u \in W_4^1(0,1)$ for which $u(0) = u(1) = 0$ and $J[u] = 0$.

- So the problem does not have a solution in $W_{4,0}^1(0,1)$.

# §17. Relaxation theory

# Relaxation Theory

- We examine the case where the functional $J$ is not weakly l.s.c.
- As we have seen from Bolza counterexample, there is no hope of obtaining, in general, the existence of a minimum for $J$.
- We could, however, associate with $J$ another functional $RJ$ whose minima should be weak cluster points of minimizing sequences of $J$.
- We now briefly discuss the case where the function $\xi \to f(x, u, \xi)$ is no longer convex. We present here a way of defining a "generalized" solution of our standard minimization problem.

# Convex Envelope

## Definition (convex envelope)

Let $f : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, $f = f(x, u, \xi)$ be continuous and non-negative.

- The convex envelope of $f$, with respect to the variable $\xi$, is denoted by $f^{**}$.
- It is the largest convex function (with respect to the variable $\xi$) which is smaller than $f$.
- In other words

$$g(x, u, \xi) \leqslant f^{**}(x, u, \xi) \leqslant f(x, u, \xi), \qquad \forall (x, u, \xi) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n$$

for every convex function $g$ (more precisely, $\xi \to g(x, u, \xi)$), $g \leqslant f$.

# Relaxation:

- Recall that the problem under consideration is

$$\inf \left\{ J[u] = \int\limits_{\Omega} f(x, u(x), \nabla u(x)) dx : u \in u_0 + \overset{\circ}{W}{}^1_{p,0} \right\} = m$$

where

1. $\Omega \subset \mathbb{R}^n$ is a bounded open set with Lipschitz boundary;
2. $f : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, $f = f(x, u, \xi)$, is continuous and non-negative;
3. $u_0 \in W^1_p(\Omega)$ with $J[u_0] < \infty$.

- Let $p \geqslant 1$ and $\alpha_1$ be such that

$$0 \leqslant f(x, u, \xi) \leqslant \alpha_1 \left(1 + |u|^p + |\xi|^p\right), \qquad \forall (x, u, \xi) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n.$$

# Relaxation (cont.):

- Consider the convex envelope $f^{**}$ and assume that $f^{**}$ is continuous.
- Construct the new minimization problem

$$\inf \left\{ J^{**}[u] = \int\limits_{\Omega} f^{**}(x, u(x), \nabla u(x))dx : u \in u_0 + W^1_{p,0}(\Omega) \right\} = m^{**}.$$

## Theorem 17.1

Let $\Omega$, $f$, $u_0$ and $f^{**}$ be as above. Then

1. $m^{**} = m$;
2. for every $u \in u_0 + W^1_{p,0}(\Omega)$, there exists $u_n \in u_0 + W^1_{p,0}(\Omega)$ so that
$$\left.\begin{array}{l} u_n \to u \quad \text{in} \quad L_p, \\ J[u_n] \to J^{**}[u] \end{array}\right. \quad \text{as} \quad n \to \infty.$$
3. If, in addition, $p > 1$ and there exist $\alpha_2 > 0$, $\alpha_3 \in \mathbb{R}$ such that
$$f(x,u,\xi) \geqslant \alpha_2 |\xi|^p + \alpha_3, \qquad \forall (x,u,\xi) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n$$
then $J^{**}$ has at least one minimizer $u^{**} \in u_0 + W^1_{p,0}(\Omega)$ and the sequence can be chosen so that
$$u_n \rightharpoonup u^{**} \quad \text{in} \quad W^1_p.$$

## Remarks

- Theorem 17.1 allows one to define $u^{**}$ as a generalized minimizer of $J$, even though $J$ may have no minimizer in $W_p^1$.
- We should emphasize that, in general, unless $f$ satisfies some coercivity condition as in Statement 3 of Theorem 17.1, the convergence of the sequence $\{u_n\}$ is of the type

$$u_n \to u \quad \text{in} \quad L_p$$

and not $u_n \to u^{**}$ in $W_p^1$.

# Computation the relaxed functionals:

- For integral functionals, one possibility to compute the relaxed functionals is to use the polar and bipolar functions.
- Let $f : \mathbb{R}^n \to \mathbb{R}$. We define the polar of $f$ (also called Legendre-Fenchel transform), the function $f^* : \mathbb{R}^n \to \mathbb{R}$, as

$$f^*(\eta) = \sup_{\xi \in \mathbb{R}^n} \{\eta \cdot \xi - f(\xi)\},$$

  and the bipolar of $f$ as

$$f^{**}(\xi) = \sup_{\eta \in \mathbb{R}^n} \{\eta \cdot \xi - f^*(\eta)\}$$

- From convex analysis it follows that $f^{**}$ is the convex envelope of $f$, i.e., the greatest convex function less than $f$.

## Example 17.1

Let $n = 1$ and $f(x) = \dfrac{|x|^p}{p}$, where $1 < p < \infty$.

Then we find

$$f^*(\eta) = \frac{|\eta|^q}{q},$$

where $q$ is, as usual, defined by $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

## Example 17.2

Let $n = 1$ and $f(x) = (x^2 - 1)^2$. We then have

$$f^{**}(x) = \begin{cases} (x^2 - 1)^2 & \text{if} \quad |x| \geqslant 1, \\ 0 & \text{if} \quad |x| < 1. \end{cases}$$

## Example 17.3

- Let us return to Bolza example.
- Here we have $n = 1$

$$f(x, u, \xi) = f(u, \xi) = (\xi^2 - 1)^2 + u^4$$

$$\inf \left\{ J[u] = \int_0^1 f(u(x), u'(x))dx : u \in W^1_{4,0}(0, 1) \right\} = m.$$

- We have already shown that $m = 0$ and that $J$ has no minimizer.
- An elementary computation (cf. Example 17.2) shows that

$$f^{**}(u, \xi) = \begin{cases} f(u, \xi) & \text{if} \quad |\xi| \geqslant 1 \\ u^4 & \text{if} \quad |\xi| < 1. \end{cases}$$

## Example 17.3 (cont.)

- Therefore $u^{**} \equiv 0$ is a solution of the minimization problem

$$\inf \left\{ J^{**}[u] = \int_0^1 f^{**}(u(x), u'(x))dx : u \in W^1_{4,0}(0,1) \right\} = m^{**} = 0.$$

- The sequence

$$u_n(x) = \begin{cases} x - \dfrac{k}{n} & \text{if} \quad x \in \left[\dfrac{2k}{2n}, \dfrac{2k+1}{2n}\right] \\ -x + \dfrac{k+1}{n} & \text{if} \quad x \in \left(\dfrac{2k+1}{2n}, \dfrac{2k+2}{2n}\right] \end{cases}.$$

satisfies the conclusions of Theorem 17.1, i.e.

$$u_n \rightharpoonup u^{**} \quad \text{in} \quad W^1_4 \quad \text{and} \quad J[u_n] \to J^{**}[u^{**}] = 0 \quad \text{as} \quad n \to \infty.$$