

Calculus of Variations

Summer Term 2017

Lecture 4

Universität des Saarlandes

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Purpose of Lesson:

- To discuss the generalization of the E-L equation to case of several variables.
- To illustrate the correspondence between the multivariable variational problems and PDEs.
- To consider a class of problems in which the functionals are required to conform with certain restrictions that are added to the usual continuity requirements and possible end-points conditions.

§4. The Case of Several Variables

The Case of Several Variables

In this subsection we consider functionals of the type

$$J[u] = \int_{\Omega} F(x, u(x), Du(x)) dx,$$

where

- Ω is a bounded open set in \mathbb{R}^n ,
- $u : \bar{\Omega} \rightarrow \mathbb{R}$ is a function of class C^2 ,
- $F(x, u, p)$ is a real valued function of class C^2 with respect of all its arguments.

Remark

- The integrand $F(x, u, p)$ is denoted as **Lagrangian**, or **variational integrand**, or **Lagrange function**.

- Consequently, the function

$$\Phi(\varepsilon) := J[u + \varepsilon\eta]$$

is defined for each $\eta \in C_0^\infty(\Omega, \mathbb{R})$ and for sufficiently small ε .

- Moreover, Φ is of class C^2 on some interval $(-\varepsilon_0, \varepsilon_0)$.
- We will call its derivative $\Phi'(0)$ at $\varepsilon = 0$ the **first variation of J** at u in direction of η and denote (sometimes)

$$\delta J[u, \eta] = \Phi'(0).$$

- A stright-forward computation yields

$$\begin{aligned} \delta J[u, \eta] = \Phi'(0) &= \int_{\Omega} \{F_u(x, u, Du)\eta + F_p(x, u, Du) \cdot D\eta\} dx \\ &= \int_{\Omega} \left\{ F_u(x, u, Du)\eta + \sum_{j=1}^n \frac{\partial F(x, u, Du)}{\partial p_j} D_j \eta \right\} dx \end{aligned}$$

- Involving the integration by parts we end up with

$$\begin{aligned}
 0 &= F_u(x, u, Du) - \sum_{j=1}^n \frac{d}{dx_j} \left[\frac{\partial F(x, u, Du)}{\partial p_j} \right] \\
 &= F_u - \sum_{j=1}^n \left[\frac{\partial^2 F}{\partial p_j \partial x_j} + \frac{\partial^2 F}{\partial p_j \partial u} D_j u + \sum_{k=1}^n \frac{\partial^2 F}{\partial p_j \partial p_k} D_k D_j u \right]
 \end{aligned}$$

Correspondence between Variational Integrals and PDEs

Example 4.1 (The Laplace equation)

Consider the *Dirichlet integral* defined by

$$\mathcal{D}[u] := \frac{1}{2} \int_{\Omega} |Du|^2 dx$$

The corresponding Euler-Lagrange equation has the form

$$\Delta u = 0 \quad \text{in } \Omega.$$

Example 4.2 (The Poisson equation)

The integral

$$J[u] = \int_{\Omega} \left[\frac{1}{2} |Du|^2 + f(x)u \right] dx$$

with the Lagrangian

$$F(x, u, p) = \frac{1}{2} |p|^2 + f(x)u$$

has the so-called *The Poisson equation*

$$\Delta u = f(x) \quad \text{in } \Omega$$

as the corresponding E-L equation.

Example 4.3 (The nonlinear Poisson equation)

The nonlinear Poisson equation

$$\Delta u = f(u) \quad \text{in } \Omega$$

is the Euler-Lagrange equation of the integral

$$J[u] = \int_{\Omega} \left\{ \frac{1}{2} |Du|^2 + g(u) \right\} dx$$

where g is a primitive function of f , i.e., $g'(z) = f(z)$.

Example 4.4 (The wave equation)

Interpret \mathbb{R}^4 as space-time continuum of points $(x, t) \in \mathbb{R}^3 \times \mathbb{R}$ where x is the position vector in \mathbb{R}^3 and t denotes the time.

Let Ω be some bounded domain in \mathbb{R}^4 , and consider the function $u(x, t)$ of x and t . Then the integral

$$J[u] = \int_{\Omega} \frac{1}{2} \left\{ u_t^2 - |Du|^2 \right\} dxdt$$

has the wave equation

$$\square u := u_{tt} - \Delta u = 0 \quad \text{in } \Omega$$

as the Euler-Lagrange equation.

Example 4.5 (The minimal surface equation)

The *area functional*

$$\mathcal{A}[u] = \int_{\Omega} \sqrt{1 + |Du|^2} dx$$

for hypersurfaces $z = u(x)$, $x \in \Omega \subset \mathbb{R}^n$, in \mathbb{R}^{n+1} yields the minimal surface equation

$$\operatorname{div} Tu = 0 \quad \text{in } \Omega, \quad Tu := \frac{Du}{\sqrt{1 + |Du|^2}}$$

as the Euler-Lagrange equation, which we can also write as

$$\sum_{i=1}^n D_i \left(\frac{D_i u}{\sqrt{1 + |Du|^2}} \right) = 0.$$

Example 4.6 (Digital Image Processing)

We consider a distorted Black-White-Image which is described by a function $f : \mathbb{R}^2 \supset \Omega \rightarrow \mathbb{R}$, here $\Omega = [0, a] \times [0, b]$.

The value $f(x)$ corresponds the darkness of gray color in the image point x (the brighter corresponds the greater value of f). In order to eliminate the interference as much as possible we minimize the functional

$$E_f[u] := \int_{\Omega} \left\{ (u - f)^2 + \alpha |Du|^2 \right\} dx, \quad \alpha > 0.$$

Here $(f - u)^2$ stands for the difference with the original image and $|Du|$ is a measure of the smoothness of the denoised image. The Integral $E_f[u]$ has the following E-L equation

$$u - \Delta u = f \quad \text{in } \Omega.$$

§5. Isoperimetric problems

We now include **additional constraints** into our minimization problems:

- Integral constraints of the form

$$\int G(x, y, y') dx = \text{const}$$

e.g., the Isoperimetric problem.

- Holonomic constraints, e.g., $G(x, y) = 0$
- Non-holonomic constraints, e.g., $G(x, y, y') = 0$
- We won't consider inequality constraints until later.

The standard example of a problem with **integral constraints** is *Dido's problem*.

Dido's problem

- This is probably one of the oldest problem in the Calculus of Variations.
- Dido (Carthaginian queen) founded the city of Carthage, in Tunisia.
- According to legend, she arrived at the site with her entourage, a refugee from a power struggle with her brother in Tyre in the Lebanon.
- She asked the locals for as much land as could be bound by a bull's hide.
- She cut the hide into a long thin strip and bounded the maximum possible area with this.



Dido Purchases Land for the Foundation of Carthage. Engraving by Matthäus Merian the Elder, in *Historische Chronica*, Frankfurt a.M., 1630. Dido's people cut the hide of an ox into thin strips and try to enclose a maximal domain.

Dido's problem falls into the class of **isoperimetric** problems.

- **iso-** (from same) and **perimetric** (from perimeter), roughly meaning "same perimeter".
- In general, such problems involve a constraint
 - e.g., the length of the bull's hide strip.
 - But the constraints is not always to fix the perimeter length.
 - Sometimes the constraint does not even involve a length.
 - But the term isoperimetric is still used.

We can write the isoperimetric problems in the following form:

The simple isoperimetric problem:

We are looking for the extremals of the functional

$$J[y] = \int_a^b F(x, y, y') dx \rightarrow \min$$

with all the usual conditions (e.g. on end points, and continuous derivatives) but in addition we must satisfy the extra functional constraint

$$G[y] = \int_a^b G(x, y, y') dx = L$$

A simplified form of Dido's problem:

Imagine that the two end-points are fixed, along the coast (Carthage was a great sea power), and we wish to encompass the largest possible area inland with a fixed length L .

We can write this problem as maximize the area

$$J[y] = \int_a^b y dx \rightarrow \max$$

encompassed by the curve y , such that the curve y has the fixed length L , e.g., as before the length of the curve is

$$G[y] = \int_a^b \sqrt{1 + y'^2} dx = L$$

subject to the end-point conditions $y(a) = y(b) = 0$.

A simplified form of Dido's problem:

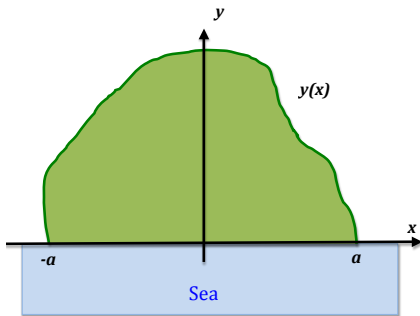
$$J[y] = \int_{-a}^a y dx \rightarrow \max$$

subject to

$$G[y] = \int_{-a}^a \sqrt{1 + (y')^2} dx = L$$

and

$$y(-a) = y(a) = 0$$



For simplicity take

$$2a < L \leq \pi a$$