Calculus of Variations
Summer Term 2017

Lecture 5

Universität des Saarlandes

05 May 2017

Purpose of Lesson:

- To consider Dido's problem.
- To discuss the catenary problem.

A simplified form of Dido's problem:

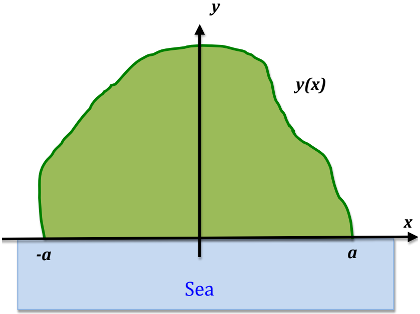
$$J[y] = \int_{-a}^{a} y\,dx \longrightarrow \max$$

subject to

$$G[y] = \int_{-a}^{a} \sqrt{1+(y')^2}\,dx = L$$

and

$$y(-a) = y(a) = 0$$



For simplicity take

$$2a < L \le \pi a$$

Approach

As before

- we perturb the curve, and consider the first variation
- but we cannot perturb by an arbitrary function $\varepsilon\eta$. because then the constraint
$$\mathcal{G}[y + \varepsilon\eta] = L$$
might be violated.
- **solution:** use the same approach as in constrained maximization, e.g. use Lagrange multipliers

Problem

To find the minimum (or maximum) of $f(x)$ for $x \in \mathbb{R}^n$ subject the constraints

$$g_i(x) = 0, \qquad i = 1, \dots, m < n \tag{5.1}$$

- Solution requires Lagrange Multipliers.
- Minimize (ot maximize) a new function (of $m + n$ variables)

$$h(x, \lambda) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x),$$

  where λ_i are the undetermined Lagrange multipliers.
- The constants $\lambda_1, \dots, \lambda_m$ are evaluated by means of the set of equations consisting of (5.1) and

$$\frac{\partial h(x, \lambda)}{\partial x_j} = 0, \qquad j = 1, \dots, n$$

Why Lagrange multipliers?

- Maximize $f(x)$ subject to $g(x) = 0$

$$h(x) = f(x) + \lambda g(x).$$

  So $\dfrac{\partial h}{\partial x_i} = 0$ implies $\dfrac{\partial f}{\partial x_i} = -\lambda \dfrac{\partial g}{\partial x_i}$.

- Assume x is an extremum which satisfies the constraint, and consider all of the $x + \Delta x$ in the neighborhood of x that also satisfy the constraint (i.e. $g(x + \Delta x) = g(x) = 0$).

- We also know from Taylor's theorem that

$$g(x + \Delta x) = g(x) + \nabla g(x) \cdot \Delta x + O(\Delta x^2)$$

  which implies that for small Δx

$$\nabla g(x) \cdot \Delta x = 0.$$

- If we take $\dfrac{\partial f}{\partial x_j} = -\lambda \dfrac{\partial g}{\partial x_j}$ then $\nabla f \cdot \Delta x = 0$.

Lagrange multipliers in functionals

To maximize

$$J[y] = \int_a^b F(x, y, y')dx$$

subject to

$$\mathcal{G}[y] = \int_a^b G(x, y, y')dx = L$$

we instead consider the problem of finding extremals of

$$\mathcal{H}[y] = \int_a^b H(x, y, y')dx = \int_a^b \left\{F(x, y, y') + \lambda G(x, y, y')\right\} dx$$

The Euler-Lagrange equations

The Euler-Lagrange equations become

$$\boxed{\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) = 0}$$

where $H = F + \lambda G$, and λ is the unknown Lagrange multiplier.

Example 5.1 (Simple Dido's problem)

$$\mathcal{H}[y] = \int_{-a}^{a} \left(y + \lambda\sqrt{1 + (y')^2} \right) dx$$

so

$$\frac{\partial H}{\partial y} = 1$$

$$\frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) = \frac{d}{dx}\left(\frac{\lambda y'}{\sqrt{1 + (y')^2}}\right)$$

and the Euler-Lagrange equation is

$$\frac{d}{dx}\frac{\lambda y'}{\sqrt{1 + (y')^2}} = 1$$

Example 5.1 (Simple Dido's problem)

- Integrating with respect to x we get

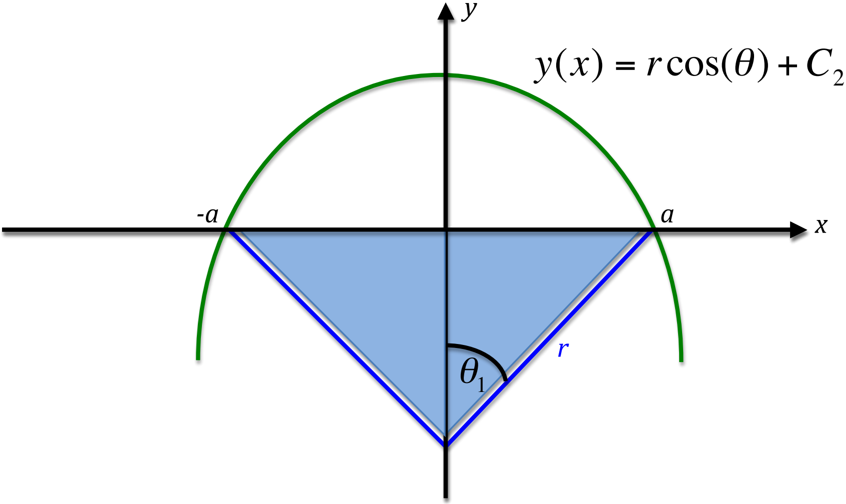
$$\begin{aligned} x + C_1 &= \lambda \sin(\theta) \\ y &= -\lambda \cos(\theta) + C_2 \end{aligned}$$

  where λ, C_1 and C_2 are determined by the two end-points, and the length of the curve L.

- We may draw a sketch of the solution, and clearly we can identify $-\lambda = r$ the radius of a circle, of which our region is a segment.

- Note we deliberately started with

$$2a < L \leqslant \pi a.$$


y
$y(x) = r\cos(\theta) + C_2$
-a
a
x
r
θ_1

Example 5.1 (Simple Dido's problem)

- We can see that the arc length of the enclosing curve will be

$$L = 2\theta_1 r$$

and the the value on the right-end determines that

$$r = \frac{a}{\sin(\theta_1)}$$

- Therefore, we have

$$L = \frac{2a\theta_1}{\sin(\theta_1)}$$

from which we may determine θ_1.

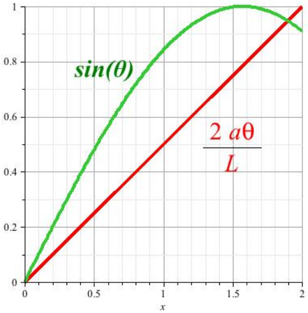
Example 5.1 (Simple Dido's problem)

- Since we determine θ_1 from

$$\sin(\theta_1) = \frac{2a}{L}\theta_1$$

we may compute

$$r = \frac{a}{\sin(\theta_1)}.$$



Example 5.1 (Simple Dido's problem)

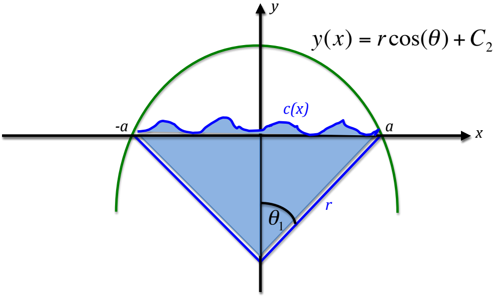
- From the conditions $y(\pm a) = 0$ it follows that

$$C_2 = -a \cot(\theta_1).$$

- The maximum possible area bounded by a curve of fixed length is a circle. So the city of Carthage is circular in shape.
- The story of Carthage isn't quite true (see picture below).

What effect would a realistic coastline have?

- Coast $c(x)$.



- Area $= \int_{-a}^{a} (y - c)\, dx$
- Note that c doesn't depend on y or y', so the Euler-Lagrange equations are unchanged, provided $c(x) < y(x)$ for the extremal.

What effect would a realistic coastline have?

- If the condition $c(x) < y(x)$ is not satisfied then the area integral includes negative components, so the problem we are maximizing is not really Dido's problem any more (she can't own negative areas).

- We really want to maximize

$$\text{Area} = \int_a^b [y - c]^+ \, dx$$

where

$$[x]^+ = \begin{cases} x, & \text{for } x > 0 \\ 0, & \text{otherwise} \end{cases}$$

- Note that the function $[x]^+$ does not have a derivative at $x = 0$.

Example 5.2 (Catenary of fixed length)

Picture: A hanging chain forms a catenary

- What happens to the shape of the suspended wire when we fix the length of the wire?

Example 5.2 (Catenary of fixed length)

- As before we seek a minimum for the potential energy

$$J[y] = \int_{x_0}^{x_1} y\sqrt{1 + (y')^2}dx \to \min$$

but now we include the constraint that the lentgh of the wire is L, e.g.

$$\mathcal{G}[y] = \int_{x_0}^{x_1} \sqrt{1 + (y')^2}dx = L$$

- We seek extremals of the new functional

$$\mathcal{H}[y] = \int_{x_0}^{x_1} (y+\lambda)\sqrt{1+(y')^2}\,dx.$$

- Notice that $H(x, y, y') = (y+\lambda)\sqrt{1+(y')^2}$ has no explicit dependence on x, and so we may compute

$$H - y'H_{y'} = \frac{(y+\lambda)(y')^2}{\sqrt{1+(y')^2}} - (y+\lambda)\sqrt{1+(y')^2} = const$$

- Perform the change of variables $u = y + \lambda$, and note that $u' = y'$ so that the above can be rewritten as

$$\frac{u(u')^2}{\sqrt{1+(u')^2}} - u\sqrt{1+(u')^2} = C_1. \tag{5.2}$$

- It is easy to see that Eq. (5.2) reduces to

$$\frac{u^2}{1+(u')^2} = C_1^2. \tag{5.3}$$

- Eq. (5.3) is exactly the same equation (in u) as we had previously for the catenary in y. So, the result is a catenary also, but shifted up or down by an amount such that the length of the wire is L.

$$y = u - \lambda = C_1 \cosh\left(\frac{x - C_2}{C_1}\right) - \lambda$$

- So, we have three constants to determine
  1. we have two end points
  2. we have the length constraint

- We put $\boxed{C_2 = 0}$ and consider the even solution with $x_0 = -1$, $y(x_0) = 1$, $x_1 = 1$ and $y(1) = 1$.
-

$$L = \int_{-1}^{1} \sqrt{1 + (y')^2} dx = \int_{-1}^{1} \cosh\left(\frac{x}{C_1}\right) dx$$

$$= C_1 \left[\sinh\left(\frac{x}{C_1}\right)\right]_{-1}^{1} = 2C_1 \sinh\left(\frac{1}{C_1}\right)$$

- Now we can calculate C_1 from the above equality.
- Once we know C_1 we can calculate λ to satisfy the end heights $y(-1) = y(1) = 1$.

- Thus, we know that a solution of the catenary problem with length constraint has the form

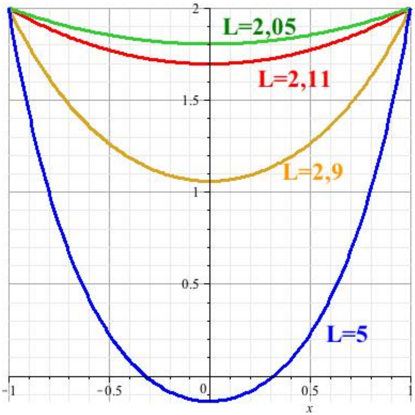
$$y = C cosh\left(\frac{x}{C}\right) - \lambda,$$

and y satisfy the additional conditions

$$y(-1) = y(1) = 2, \qquad L = \int\limits_{-1}^{1} \sqrt{1 + (y')^2} dx = 2C \sinh\left(\frac{1}{C}\right).$$

- Using Maple we calculate y for the natural catenary (without length constraint), as well as for $L = 2.05$ $L = 2.9$ and $L = 5$. See Worksheet 1 for the detailed calculation.

- All catenaries are valid, but one is natural



- The red curve shows the natural catenary (without length constraint), and the green, yellow and blue curves show other catenaries with different lengths.

Pathologies

Note that in both cases (”simple Dido’s problem” and ”catenary of fixed length”)

- the approach only works for certain ranges of L.
- If L is too small, there is no physically possible solution
  - e.g., if wire length $L < x_1 - x_0$
  - e.g., if oxhide length $L < x_1 - x_0$
- If L is too large in comparison to $y_1 = y(x_1)$, the solution may have our wire dragging on the ground.