Calculus of Variations
Summer Term 2017

Lecture 6

Universität des Saarlandes

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Purpose of Lesson:

- Consider rigid extremals and give interpretation of the Lagrange multiplier λ
- To solve the more general case of Dido's problem with general shape and parametrically described perimeter.
- To discuss why does the Lagrange multiplier approach work for functionals.

Rigid Extremals

A particular problem to watch for are rigid extremals

- Rigid extremals are extremals that cannot be perturbed, and stiil satisfy the constraint.

Example 5.3

- For example

$$\mathcal{G}[y] = \int_0^1 \sqrt{1 + (y')^2} dx = \sqrt{2}$$

  with the boundary constraints $y(0) = 0$ and $y(1) = 1$.

- The only possible y to satisfy this constraint is $y(x) = x$, so we cannot perturb around this curve to find conditions for viable extremals.

- Rigid extremals cases have some similarities to maximization of a function, where the constraints specify a single point:

Example 5.4

Maximize $f(x, y) = x + y$, under the constraint that $x^2 + y^2 = 0$.

Interpretation of λ:

Consider again to finding extremals for

$$\mathcal{H}[y] = J[y] + \lambda\mathcal{G}[y], \tag{5.4}$$

where we include $\mathcal{G}$ to meet an isoperimetric constraint $\mathcal{G}[y] = L$.

- One way to think about λ is to think of (5.4) as trying to minimize $J[y]$ and $\mathcal{G}[y] - L$.
  1. λ is a tradeoff between J and $\mathcal{G}$.
  2. If λ is big, we give a lot of weight to $\mathcal{G}$.
  3. If λ is small, then we give most weight to J.
- So, λ might be thought of as how hard we have to ”pull” towards the constraint in order to make it.

Interpretation of λ (cont.)

For example,

- in the catenary problem, the size of λ is the amount we have to shift the cosh function up or down to get the right length.
- when $\lambda = 0$ we get the natural catenary,

  i.e., in this case, we didn't need to change anything to get the right shape, so the constraint had no affect.

Interpretation of λ (cont.)

Write the problem (including the constant) as minimize

$$\mathcal{H}[y] = \int F + \lambda(G - k)dx,$$

for the constant $k = \dfrac{L}{\int 1 dx}$, then

$$\frac{\partial \mathcal{H}}{\partial k} = -\lambda,$$

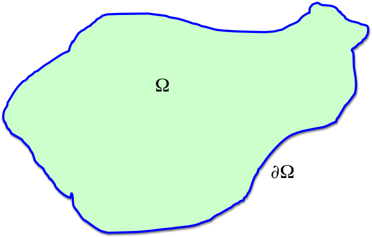
- we can also think of λ as the rate of change of the value of the optimum with respect to k.
- when $\lambda = 0$, the functional $\mathcal{H}$ has a stationary point

  e.g., in the catenary problem this is a local minimum corresponding to the natural catenary.

Dido's problem - general case

Consider now the more general case of Dido's problem:

- a general shape,



- without a coast,

so that the perimeter must be parametrically described.

Dido's problem is usual posed as follows:

Problem 5-1 (Dido's problem -traditional)

To find the curve of length L which encloses the largest possible area, i.e., maximize

$$\text{Area } = \iint\limits_{\Omega} 1 dxdy$$

subject to the constraint

$$\oint\limits_{\partial\Omega} 1 ds = L$$

Of course Problem 5-1 is not yet in a convinient form.

Green's Theorem converts an integral over the area Ω to a contour integral around the boundary $\partial\Omega$.

Green's Theorem

$$\iint\limits_{\Omega} \left(\frac{\partial \phi}{\partial x} + \frac{\partial \varphi}{\partial y} \right) dxdy = \oint\limits_{\partial\Omega} \phi dy - \varphi dx$$

for $\phi, \varphi : \Omega \to \mathbb{R}$ such that ϕ, φ, ϕ_x and φ_y are continuous.

This converts an area integral over a region into a line integral around the boundary.

- The area of a region is given by

$$\text{Area} = \iint_{\Omega} 1 dxdy.$$

- In Green's theorem choose $\phi = \dfrac{x}{2}$ and $\varphi = \dfrac{y}{2}$, so that we get

$$\text{Area} = \iint_{\Omega} 1 dxdy = \frac{1}{2} \oint_{\partial\Omega} xdy - ydx$$

- Previous approach to Dido, was to use $y = y(x)$, but in more general case where the boundary must be closed, we can't define y as a function of x (or visa versa).

- So, we write the boundary curve parametrically as $(x(t), y(t))$.

- If the boundary $\partial\Omega$ is represented parametrically by $(x(t), y(t))$ then

$$\text{Area} = \iint_{\Omega} 1\,dxdy = \frac{1}{2}\oint_{\partial\Omega} xdy - ydx = \frac{1}{2}\oint_{\partial\Omega} (x\dot{y} - y\dot{x})\,dt$$

- So, now the problem is written in terms of

$$\begin{aligned} \text{one independent variable } &= t \\ \text{two dependent variables } &= (x, y). \end{aligned}$$

- Previously we wrote the isoperimetric constraint as

$$\mathcal{G}[y] = \int 1 ds = \int_{x_0}^{x^1} \sqrt{1 + (y')^2} dx = L$$

- Now we must also modify the constraint using

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

to get

$$\mathcal{G}[y] = \oint 1 ds = \oint \sqrt{\dot{x}^2 + \dot{y}^2} dt = L$$

- Hence, we look for extremals of

$$\mathcal{H}[x,y] = \oint \left(\frac{1}{2}\left(x\dot{y} - y\dot{x}\right) + \lambda\sqrt{\dot{x}^2 + \dot{y}^2} \right) dt$$

- So, $H(t,x,y,\dot{x},\dot{y}) = \frac{1}{2}\left(x\dot{y} - y\dot{x}\right) + \lambda\sqrt{\dot{x}^2 + \dot{y}^2}$, and there are two dependent variables, with derivatives

$$\frac{\partial H}{\partial x} = \frac{1}{2}\dot{y} \qquad \frac{\partial H}{\partial \dot{x}} = -\frac{1}{2}y + \frac{\lambda\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}}$$

$$\frac{\partial H}{\partial y} = -\frac{1}{2}\dot{x} \qquad \frac{\partial H}{\partial \dot{y}} = \frac{1}{2}x + \frac{\lambda\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}$$

- Leading to the 2 Euler-Lagrange equations

$$\frac{d}{dt}\left[-\frac{1}{2}y + \frac{\lambda \dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right] = \frac{1}{2}\dot{y}$$

$$\frac{d}{dt}\left[\frac{1}{2}x + \frac{\lambda \dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right] = -\frac{1}{2}\dot{x}$$

- Integrate

$$-\frac{1}{2}y + \frac{\lambda \dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}} = \frac{1}{2}y + A$$

$$\frac{1}{2}x + \frac{\lambda \dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}} = -\frac{1}{2}x - B$$

- After simplification we get

$$\frac{\lambda \dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = y + A$$
$$\frac{\lambda \dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = -x - B$$

- Now square the both equations, and add them to get

$$\lambda^2 \frac{\dot{x}^2 + \dot{y}^2}{\dot{x}^2 + \dot{y}^2} = (y + A)^2 + (x + B)^2$$

- Or, more simply

$$(y + A)^2 + (x + B)^2 = \lambda^2,$$

the equation is a circle with center $(-B, -A)$ and radius $|\lambda|$.

Remarks

- Note, we can't set value at end points arbitrarily.
- If $x(t_0) = x(t_1)$, and $y(t_0) = y(t_1)$, then we get a closed curve, obviously a circle.
- These conditions only amount to setting one constant, λ.
- On the other hand, if we specify different end-points, we are really solving a problem such as the simplified problem considered in Lecture 5.

Why the Lagrange multiplier approach works for functionals?

- Consider the approximation of the functional

$$J[y] = \int_a^b F(x, y, y')dx \simeq \sum_{i=1}^{n} F\left(x_i, y_i, \frac{\Delta y_i}{\Delta x_i}\right) \Delta x = F(y_1, \dots, y_n)$$

  where $\Delta x = \frac{(b-a)}{n}$, and $\Delta y_i = y_i - y_{i-1}$.

- The problem of finding an extremal curve now becomes one of finding stationary points of the function $F(y_1, \dots, y_n)$.

- We solve this by looking for

$$\frac{\partial F}{\partial y_i} = 0 \quad \text{for all} \quad i = 1, 2, \dots, n.$$

- The constraint can be likewise approximted to give

$$\mathcal{G}[y] \simeq \sum_{i=1}^{n} G\left(x_i, y_i, \frac{\Delta y_i}{\Delta x_i}\right) \Delta x = G(y_1, \dots, y_n) = L.$$

- Under our usual conditions on J and $\mathcal{G}$, the limit as $n \to \infty$ gives

$$F(y_1, \dots, y_n) \to J[y]$$
$$G(y_1, \dots, y_n) \to \mathcal{G}[y]$$

- That is, the functions of the approximation $y_1, \dots, y_n$ converge to the functionals of the curve $y(x)$.

- In the finite dimensional case the constraint is

$$G(y_1, \dots, y_n) - L = 0$$

and we use a standard Lagrange multiplier

$$H(y_1, \dots, y_n, \lambda) = F(y_1, \dots, y_n) + \lambda \left[G(y_1, \dots, y_n) - L \right]$$

- We solve this by looking for

$$\frac{\partial H}{\partial y_i} = 0, \quad \forall i = 1, 2, \dots, n, \quad \text{and} \quad \frac{\partial H}{\partial \lambda} = 0.$$

- The last equation just gives you back your constraint.

- In our formulation of the isoperimetric problem we take

$$\mathcal{H}[y] = J[y] + \lambda \mathcal{G}[y]$$

  and we also have

$$H(y_1, \dots, y_n, \lambda) = F(y_1, \dots, y_n) + \lambda \left[G(y_1, \dots, y_n) - L \right].$$

- In the limit as $n \to \infty$ we find that

$$H(y_1, \dots, y_n, \lambda) \to \mathcal{H}[y] - \lambda L.$$

- The EL equations for $\mathcal{H}[y] - \lambda L$ and $\mathcal{H}[y]$ are the same, so they have the same extremals.

Remarks about multiple constraints

- We can also handle multiple constraints via multiple Lagrange multipliers.
- For instance, if we wish to find extremals of $J[y] = \int\limits_{x_0}^{x_1} F(x, y, y')dx$

  with the m constraints

$$\mathcal{G}_k[y] = \int\limits_{x_0}^{x_1} G_k(x, y, y')dx = L_k$$

  we would look for extremals of

$$\mathcal{H}[y] = \int\limits_{x_0}^{x_1} H(x, y, y')dx = \int\limits_{x_0}^{x_1} \left[F(x, y, y') + \sum_{k=1}^{m} \lambda_k G_k(x, y, y') \right] dx$$