

Calculus of Variations

Summer Term 2017

Lecture 9

Universität des Saarlandes

19 May 2017

Purpose of Lesson:

- To discuss necessary and sufficient conditions for extrema
- To discuss the Legendre condition

§8. Classification of extrema

Introductory remarks

- We have so far typically ignored the issue of classification of extrema.
- Contrariwise, we remember that for simple stationary points we need to look of higher derivatives to see if a stationary point is a maximum, minimum or point of inflection.
- We need an analogous process for extremal curves as well.

Let $f : \mathbb{R} \rightarrow \mathbb{R}$. Local extrema have $f'(x) = 0$.

- $f''(x) > 0$ local **minima**
- $f''(x) < 0$ local **maxima**
- $f''(x) = 0$ it might be a stationary **point of inflection**, depending on higher order derivatives. More precisely, let f be a real-valued, sufficient differentiable function on the interval $I \subset \mathbb{R}$, $x \in I$ and $n \geq 1$ an integer.

If now holds

$$f'(x) = f''(x) = \dots = f^n(x) = 0 \quad \text{and} \quad f^{n+1}(x) \neq 0$$

then, either

- n is odd and we have a local extremum at x

or

- n is even and we have a (local) saddle point at x .

- The E-L equation is a **necessary** condition.
- The E-L equation is **not sufficient**.
- Along the extremal curve the functional might have
 - a min, max or stationary point
 - it might be global or local
- We really need to classify extremals
 - Until now we have
 - just assumed it was the minima
 - used analytical insight to understand the solution
 - tested it by inspection
 - We could also compare to alternative curves.

Examples

- **Physical intuition:** Brachystochrone (or geodesic): we look for a minimum time path. So, we can see that **physically** there can't be a maximum.
- **Examine the solution:** e.g., consider the functional

$$J[y] = \int_0^1 y'^2 dx$$

conditioned on $y(0) = y(1) = 0$.

The E-L equation gives straight line solutions, e.g., $y = c_1 x + c_2$, and the BCs imply $c_1 = c_2 = 0$, so $y' = 0$. Clearly then $J[y] = 0$, which is the minimum possible value, for an integral of a non-negative function like y'^2 .

Examples (cont.)

- **Compare with alternative curves:** Consider the functional

$$J[y] = \int_0^1 (xy' + y'^2) dx$$

conditioned on $y(0) = 0$ and $y(1) = 1$.

The E-L equation gives

$$y = -\frac{1}{4}x^2 + C_1x + C_2$$

and the BCs give $C_1 = 5/4$ and $C_2 = 0$, so the solution is

$$y = \frac{5}{4}x - \frac{1}{4}x^2.$$

Examples (cont.)

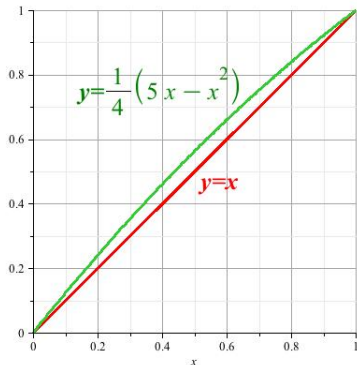
For $y = \frac{5}{4}x - \frac{1}{4}x^2$, we have $y' = \frac{5}{4} - \frac{1}{2}x$, and, consequently,

$$\begin{aligned} J[y] &= \int_0^1 \left[x \left(\frac{5}{4} - \frac{1}{2}x \right) + \left(\frac{5}{4} - \frac{1}{2}x \right)^2 \right] dx \\ &= \int_0^1 \left[\frac{25}{16} - \frac{1}{4}x^2 \right] dx \\ &= \left[\frac{25}{16}x - \frac{1}{12}x^3 \right]_0^1 \\ &= \boxed{\frac{71}{48}}. \end{aligned}$$

Examples (cont.)

For the curve $y(x) = x$, we have $y' = 1$, and so the functional is

$$J[y] = \int_0^1 (x+1)^2 dx = \left[\frac{x^3}{3} + x^2 + x \right]_0^1 = \boxed{\frac{7}{3}}.$$



Examples (cont.)

Now,

$$\frac{7}{3} > \frac{71}{48},$$

so, we should be looking at a local min.

But, it isn't very formal, or rigorous!

Classification of extrema

- All the methods listed above either
 - Aren't very formal or rigorous
 - Aren't easy to generalize
- Need to develop a means of formal classification
- The secret is by analogy to classification for functions of several variables
 - We need to look at second derivatives
 - Positive definiteness of the Hessian
- The analogy to second derivatives is called the **second variation**.

Maxima of n variables

- If a smooth function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ has a local extrema at $\mathbf{x} = (x_1, x_2, \dots, x_n)$ then $\nabla f(\mathbf{x}) = 0$.
- So, we can rewrite Taylor's theorem for small $\mathbf{h} = (h_1, h_2, \dots, h_n)$ as

$$f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}) \approx \frac{1}{2} \mathbf{h}^T H(\mathbf{x}) \mathbf{h}$$

- A sufficient condition for the extrema \mathbf{x} to be a local minimum is for the quadratic form

$$Q(h_1, \dots, h_n) = \mathbf{h}^T H(\mathbf{x}) \mathbf{h} = \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j} h_i h_j$$

to be strictly positive definite.

Definition 8.1

- A quadratic form

$$Q(\mathbf{x}) = \sum_{i,j} a_{ij}x_i x_j = \mathbf{x}^T \mathbf{A} \mathbf{x}$$

is said to be positive definite if $Q(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{0}$.

- A quadratic form Q is positive definite iff every eigenvalue of A is greater than zero.
- A quadratic form Q is positive definite if all the principal minors in the top-left corner of A are positive, in other words

$$a_{11} > 0, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \quad \dots$$

Notes on maxima and minima

- Maxima of $f(\mathbf{x})$ is minima of $-f(\mathbf{x})$.
- We need to generalize this for functionals.
- We do this using the second variation.
- Note that even so, we only classify local min and max, the global min and max may occur at the boundary, or at one of the several extrema.

The second variation

- Once again consider the fixed end-point problem, with small perturbations about the extremal curve

$$J[y] = \int_a^b F(x, y, y') dx, \quad y(a) = A, \quad y(b) = B,$$

$$\bar{y} = y + \varepsilon\eta, \quad \eta(a) = \eta(b) = 0.$$

- We take the second derivative of $\phi(\varepsilon) = J[y + \varepsilon\eta]$ with respect to ε , evaluate it $\varepsilon = 0$; that is,

$$\begin{aligned} \frac{d^2\phi(\varepsilon)}{d\varepsilon^2} &= \frac{d^2}{d\varepsilon^2} J[y + \varepsilon\eta] \Big|_{\varepsilon=0} \\ &= \int_a^b \left[\frac{\partial^2 F}{\partial y^2} \eta^2(x) + 2 \frac{\partial^2 F}{\partial y \partial y'} \eta(x) \eta'(x) + \frac{\partial^2 F}{\partial (y')^2} (\eta'(x))^2 \right] dx \end{aligned}$$

The second variation (cont.)

- Note that

$$2\eta\eta' = \frac{d}{dx} (\eta^2)$$

- So, we can write

$$\begin{aligned} \int_a^b 2\eta\eta' F_{yy'} dx &= \int_a^b \frac{d}{dx} (\eta^2) F_{yy'} dx \\ &= \left[\eta^2 F_{yy'} \right]_a^b - \int_a^b \eta^2 \frac{d}{dx} (F_{yy'}) dx \end{aligned}$$

using integration by parts and the fact that $\eta(a) = \eta(b) = 0$.

The second variation (cont.)

- Now we define the **second variation** by

$$\begin{aligned} \delta^2 J[y, \eta] &= \int_a^b \left[\frac{\partial^2 F}{\partial y^2} \eta^2(x) + 2 \frac{\partial^2 F}{\partial y \partial y'} \eta(x) \eta'(x) + \frac{\partial^2 F}{\partial (y')^2} (\eta'(x))^2 \right] dx \\ &= \int_a^b \left[\eta^2 \left(F_{yy} - \frac{d}{dx} F_{yy'} \right) + (\eta'(x))^2 F_{y'y'} \right] dx \end{aligned}$$

- This form has the advantage that
 - $\eta^2 \geq 0$
 - $(\eta'(x))^2 \geq 0$
 - after solving the Euler-Lagrange equation we know F and its derivatives.

Classifying extrema

For an extremal curve y to be a local minima, we require

$$\delta^2 J[y, \eta] \geq 0$$

for all valid perturbation curves η .

Likewise we get a maxima if $\delta^2 J[y, \eta] \leq 0$ for all η and a stationary curve if the second variation changes sign.

- Note that we have already solved the E-L equation and so we know y . Hence we can calculate F_{yy} , $F_{yy'}$ and $F_{y'y'}$ explicitly.
- We still need to ensure $\delta^2 J[y, \eta] \geq 0$ for all possible η .

Legendre condition

The **Legendre condition** is a **necessary** condition for a local minima.

The Legendre condition:

If y is a local minima of the functional $J[y] = \int F(x, y, y') dx$, then

$$F_{y'y'}(x, y, y') \geq 0$$

along the extremal curve y .

The Legendre condition (sketch of the proof):

- Remember that F and y are known functions (now), so we know F_{yy} , $F_{yy'}$ and $F_{y'y'}$, explicitly as functions of x .
- Hence we can write the second variation as

$$\delta^2 J[y, \eta] = \int_a^b \left[\eta^2 B(x) + \eta'^2 A(x) \right] dx$$

where

$$A(x) = F_{y'y'}$$

$$B(x) = \left(F_{yy} - \frac{d}{dx} F_{yy'} \right)$$

The Legendre condition (sketch of the proof):

- The proof relies on the fact that we can find functions η such that $|\eta|$ is small, but $|\eta'|$ is large.
- Note that we cannot do the opposite, because $|\eta'|$ small implies that η is smooth, which given the end conditions implies that $|\eta|$ will be small.

Example 8.1 (Mollifier)

$$\eta(x) = \begin{cases} \exp\left(-\frac{\gamma}{\gamma^2 - (x - c)^2}\right), & \text{if } x \in [c - \gamma, c + \gamma] \\ 0, & \text{otherwise} \end{cases}$$

Example 8.1 (Mollifier - cont.)

$$\eta(x) = \begin{cases} \exp\left(-\frac{\gamma}{\gamma^2 - (x - c)^2}\right), & \text{if } x \in [c - \gamma, c + \gamma] \\ 0, & \text{otherwise} \end{cases}$$

$$\eta'(x) = \begin{cases} -\frac{2\gamma(x - c)}{(\gamma^2 - (x - c)^2)^2} \exp\left(\frac{-\gamma}{\gamma^2 - (x - c)^2}\right), & \text{if } x \in [c - \gamma, c + \gamma] \\ 0, & \text{otherwise} \end{cases}$$

Ratio of derivative to function is larger for smaller γ .

The Legendre condition (sketch of the proof):

- Given $|\eta|$ small, we can essentially ignore the η^2 terms, and we get only the term

$$\delta^2 J[y, \eta] = \int_a^b \eta'^2 A(x) dx$$

- If A changes sign, then we could choose η to be a mollifier such that it is localized in the part where A is positive, and a mollifier such that it is localized in the part where A is negative.
- The two mollifiers would produce integrals with different signs, and so we would get a change of sign of $\delta^2 J[y, \eta]$, which is what we are trying to avoid.

Example 8.2

Find the minimum of the functional

$$J[y] = \int_0^1 (xy' + y'^2) dx$$

conditioned on $y(0) = 0$ and $y(1) = 1$.

The solution is

$$y = \frac{1}{4} (5x - x^2).$$

Then (from earlier)

$$F(x, y, y') = xy' + y'^2 = \frac{25}{16} - \frac{1}{4}x^2$$

Example 8.2 (cont.)

$$F(x, y, y') = xy' + y'^2$$

$$F_{y'} = x + 2y'$$

$$F_{y'y'} = 2 > 0$$

Hence Legendre's condition is satisfied, so this **could** be a local minimum.

Sufficient condition

- various approaches to sufficient conditions
- problem is that we have to get away from pointwise conditions
 - like the Legendre condition
 - pointwise conditions couldn't classify which of two possible arcs of a great circle is the shortest path between two points on a sphere.
- a sufficient condition is the Jacobi condition, but there are others
- still mostly conditions for local minima, so need to do more work

Example 8.3

Find the minimum of the functional

$$J[y] = \int_0^1 (xy' + y'^2) dx$$

So

$$F_{y'y'} = 2$$

$$F_{yy'} = F_{yy} = 0$$

Therefore the second variation

$$\delta^2 J[y, \eta] = \int_0^1 \left[\eta^2 \left(F_{yy} - \frac{d}{dx} F_{yy'} \right) + \eta'^2 F_{y'y'} \right] dx = 2 \int_0^1 \eta'^2 dx \geq 0$$

for all η we have a local minimum!