Universität des Saarlandes
Fachrichtung 6.1 – Mathematik
PD Dr. Darya Apushkinskaya

Calculus of Variations (Summer Term 2016)
Assignment H3 - Homework

Problem 3.1 (4+4=8 Points)

Find the form of extremals of the following functionals

$$\text{a)} \qquad J_1[y,z] = \int_{x_0}^{x_1} \left(2yz - 2y^2 + y'^2 - z'^2\right) dx$$

$$\text{b)} \qquad J_1[q_1,q_2,q_3] = \int_{t_1}^{t_2} \left(\dot{q}_1\dot{q}_2 + \dot{q}_2 q_3 + \dot{q}_3 q_1\right) dt$$

Problem 3.2 (9 Points)

Newton's aerodynamic problem (the problem of finding the surface of revolution that minimizes drag) is often approximated by assuming the shape is long and thin, so that y' is large (and negative). In this case approximate

$$\frac{1}{1+(y')^2} \simeq \frac{1}{(y')^2}$$

and the functional of interest by

$$J[y] \simeq \int_0^R \frac{x}{(y')^2} dx,$$

Derive the shape that arise from minimizing this functional.

Problem 3.3 (5 Points)

The Beltrami identity states that the extremal function of the integral

$$I[u] = \int_a^b L(x,u,u')dx$$

satisfies the differential equation

$$\frac{d}{dx}\left(L - u'\frac{\partial L}{\partial u'}\right) - \frac{\partial L}{\partial x} = 0.$$

Please prove the identity using the Euler-Lagrange equation and the chain rule. Note that as a special case, when L does not depend on x, we get the equation for the autonomous case, i.e., $H = const.$

Problem 3.4 (3+3=6 Points)

Find the extremals of the functionals below subject to the fixed end point conditions prescribed

a) $J_1[y] = \int_0^{\pi/2} \left(y^2 + y'^2 - 2y\sin x\right) dx, \quad y(0) = 0, \quad y(\pi/2) = 3/2$

b) $J_2[y] = \int_1^2 \frac{y'^2}{x^3} dx, \quad y(1) = 0, \quad y(2) = 15$

Deadline for submission: Tuesday, June 07, 08:30 am