Universität des Saarlandes
Fachrichtung 6.1 – Mathematik
PD Dr. Darya Apushkinskaya

Calculus of Variations (Summer Term 2017)
Assignment H5 - Homework

Problem 5.1 (9 Points)

Use Ritz's method to find an approximate, non-trivial solution to the differential equation

$$y'' + \frac{1}{x}y' + \lambda y = 0$$

in the domain $x \in [0, 1]$ where $y(0)$ is non-singular and $y(1) = 1$, and hence determine an approximate value of λ that has a solution.

(*Hint:* Note that the equation can be written in the form

$$\frac{d}{dx}(xy') + \lambda xy = 0$$

and find the corresponding integral for which this is the Euler-Lagrange equation.

Once you have a variational problem, use the trial function

$$y_{trial} = a + bx^2 + cx^4,$$

which we have chosen because the solution is expected to be an even function.)

Problem 5.2 (8 Points)

How could we minimize the functional

$$J[y] = \int_0^1 \left(y^2 + y'^2\right) dx, \qquad y(0) = 0, \quad y(1) = 1$$

by the method of Ritz?

(*Hint:* If we introduce a new function $z(t)$

$$z(t) = (1 - x)\, y(t)$$

we note that $z(t)$ satisfies $z(0) = 0$ and $z(1) = 0$.)

Problem 5.3 (8 Points)

Find the minimum value of

$$J[u] = x(1) + \int_0^1 \alpha u^2 dt,$$

where $\alpha > 0$, $x(0) = 0$, $x(1)$ free, and

$$\dot{x} = u.$$

How does the answer change if we add the condition that $|u(t)| \leqslant 1$?

Problem 5.4 (10 Points)

Maximize the range of a missile: Take a missile which has a rocket motor that generates constant thrust f for a fixed time interval $[0, t_1]$. We can control the angle of the thrust $\theta(t)$ (relative to the horizontal). Ignoring drag, the curve of the Earth's surface (and its rotation), determine the angle profile that will maximize the range of the missile.

Hints: choose a coordinates (x, y), and $(u, v) = (\dot{x}, \dot{y})$, then the DEs decribing the system under thrust will be

$$\begin{aligned} \dot{x} &= u \\ \dot{y} &= v \\ \dot{u} &= f \cos\theta \\ \dot{v} &= f \sin\theta - g \end{aligned}$$

After the rocket stops firing, the missile will continue on a ballistic trajectory, i.e., the remaining motion will be a parabola, resulting in a total firing distance of

$$R(x, y, u, v) = x + \frac{u}{g}\left[v + \sqrt{v^2 + 2gy}\right]$$

where x, y, u, v are given at the time at which ballistic motion commences.

Problem 5.5 (4 Points)

Let

$$u(x) = \begin{cases} x, & \text{if} \quad 0 \leqslant x \leqslant 1, \\ 1, & \text{if} \quad 1 < x \leqslant 2, \end{cases} \qquad \text{and} \qquad v(x) = \begin{cases} 1, & \text{if} \quad 0 \leqslant x \leqslant 1, \\ 0, & \text{if} \quad 1 < x \leqslant 2. \end{cases}$$

Show that v is a weak derivative of u.

Deadline for submission: Tuesday, July 18, 02:00 p.m.