Mathematics is a game played according to certain simple rules with meaningless marks on paper.
David Hilbert
(1866-1943, German mathematician)

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Assignment 1 for the lecture

Modeling with Partial Differential Equations

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Submission: Wednesday, 31 October 2018, until 2 p.m.

Exercise 1.1. (6 x 0.5 = 3 points)
Classify the following differential equations into partial or ordinary differential equations, linear or non-linear, homogeneous or inhomogeneous equations. Additionally, determine the order of the differential equation.

(a) $u'(t) = e^t u(t)$

(b) $u''(x) = u(x)\sqrt{x}$

(c) $u_{xx}(x,y) + u_{yy}(x,y)e^{\sin(x)} = 1$

(d) $u_t(x,t) + u_x(x,t) = u_{xx}(x,t) + u^2(x,t)$

(e) $(u'(t))^2 + u(t) = e^t$

(f) $u_x(x,y) + u_{xy}(x,y) + u_y(x,y) = \cos(u(x,y))$

Exercise 1.2. (1 + 2 + 3 = 6 points) Let $\Omega = (a,b) \subset \mathbb{R}$ be non-empty and open. With $C_0^\infty(\Omega)$ we denote the smooth and compactly supported functions in Ω. For

$$u \in L_{\text{loc}}(\Omega) = \{f : \Omega \to \mathbb{R} |\ f\,|_K \in L_1(K) \text{ for all } K \subset \Omega \text{ compact}\},$$

$w \in L_{\text{loc}}(\Omega)$ is the weak derivative of u, if

$$-\int_\Omega u\,\varphi'\,\mathrm{d}x = \int_\Omega w\,\varphi\,\mathrm{d}x \quad \text{for all } \varphi \in C_0^\infty(\Omega).$$

(a) Show that $u \in C^1(\Omega)$ is weakly differentiable with weak derivative u'.

(b) In the following, let $\Omega = \mathbb{R}$. Show that a weak derivative of

$$u : \mathbb{R} \to \mathbb{R}, x \mapsto \frac{1}{2}(1 + |x|).$$

is given by $w = \frac{1}{2}\,\text{sgn}$.

(c) Show that

$$-\int \frac{1}{2}\,\text{sgn}(x)\,\varphi'(x)\,\mathrm{d}x = \varphi(0) \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}).$$

Does $\frac{1}{2}\,\text{sgn}$ have a weak derivative?
Hint: Do a proof by contradiction and consider $\psi \in C_0^\infty(\mathbb{R} \setminus \{0\})$.

Exercise 1.3. (2 + 4 = 6 points) In this exercise, we use the following theorem and proof it with stronger assumptions:

> Let $\Omega \subset \mathbb{R}$ non-empty and open. Suppose $f \in L_{\mathrm{loc}}(\Omega)$. If
>
> $$\int_\Omega f\,\varphi\,\mathrm{d}x = 0$$
>
> for all $\varphi \in C_0^\infty(\Omega)$, then
>
> $$f = 0 \text{ a.e.}$$

(a) Use the above theorem to show that weak derivatives are unique up to sets of measure zero.

(b) Proof the theorem under the additional assumption of $f \in C(\Omega)$.
Hint: $\{x \in \Omega : f(x) > 0\} \subset \mathbb{R}$ is open.

Exercise 1.4. (2 + 3 = 5 points) Let $(V, \|\cdot\|)$ be a real normed vector space. Furthermore, let $a : V \times V \to \mathbb{R}$ be a symmetric, bounded and coercive bilinear form and $\varphi : V \to \mathbb{R}$ a linear and bounded functional. The variational formulation (VF)

$$\begin{gathered}\text{Find } u \in V \text{ such that}\\ \forall v \in V : a(u, v) = \varphi(v)\end{gathered}$$

admits a unique solution $u \in V$.

(a) Show that the solution of the VF minimises the functional

$$E : V \to \mathbb{R}, v \mapsto \frac{1}{2}a(v, v) - \varphi(v)$$

in V.

(b) Conversely, show that any minimiser of E solves the VF.