In der Mathematik versteht man die Dinge nicht.
Man gewöhnt sich nur an sie.
John von Neumann
(1903-1957, Hungarian-American mathematician)

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Assignment 3 for the lecture

Modeling with Partial Differential Equations

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Exercise 3.1. (5 points) The aim of this exercise is to show that the trace theorem does not hold if the domain is not regular. Take the open set

$$\Omega = \{(x, y) \in \mathbb{R}^2 : 0 < x < 1, 0 < y < x^r\}$$

with $r > 2$ and the function $f : \Omega \to \mathbb{R}, (x, y) \mapsto x^\alpha$, $\alpha \in \mathbb{R}$. Show that $f \in H^1(\Omega)$ if and only if $2\alpha + r > 1$, while $f \in L_2(\partial\Omega)$ if and only if $2\alpha > -1$. Conclude the result.

Exercise 3.2. (5 points) The aim of this exercise is to show that the trace operator is unbounded for L_2 functions i.e. there is no $C > 0$ such that for all $v \in L_2(\Omega)$

$$\|v\,|_{\partial\Omega}\|_{L_2(\partial\Omega)} \le C\|v\|_{L_2(\Omega)}.$$

For simplicity, we choose the set $\Omega \subset \mathbb{R}^d$ as the d-dimensional unit ball.
Hint: Construct a sequence of regular functions that are 1 on the boundary and whose $L_2(\Omega)$-norms tend to zero.

Exercise 3.3. (1 + 4 = 5 points) Let $\varphi \in C_0^\infty(\mathbb{R})$ be non-zero and define the sequence $(u_n)_{n\in\mathbb{N}}$ by

$$u_n(x) = \varphi(x + n), \quad x \in \mathbb{R}, n \in \mathbb{N}.$$

(a) Check that $(u_n)_{n\in\mathbb{N}}$ is bounded in $W_p^1(\mathbb{R})$ for $1 \le p \le \infty$.

(b) Prove that there exists no subsequence converging in $L_q(\mathbb{R})$ for any $1 \le q \le \infty$.
Hint: Show that $(u_n)_{n\in\mathbb{N}}$ has no Cauchy subsequence in $L_q(\mathbb{R})$.

Exercise 3.4. (5 points) Solve the hyperbolic equation

$$\begin{aligned} x_1\frac{\partial u}{\partial x_1} + 2x_2\frac{\partial u}{\partial x_2} + \frac{\partial u}{\partial x_3} &= 3u \quad &&\text{for } x \in \mathbb{R}^3, \\ u(x_1, x_2, 0) &= \varphi(x_1, x_2) \quad &&\text{for } (x_1, x_2)^\top \in \mathbb{R}^2, \end{aligned}$$

with the method of characteristics.