Ich verstehe nicht, dass man die Mathematik nicht versteht.
Henri Poincaré
(1854-1912, French mathematician)

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Assignment 4 for the lecture
Modeling with Partial Differential Equations
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Exercise 4.1. (1 + 3.5 + 3.5 = 8 points) We consider the differential equation

$$x_1 \frac{\partial u}{\partial x_1} - x_2 \frac{\partial u}{\partial x_2} = g(u), \quad x \in \Omega \tag{1}$$

with the boundary condition

$$u(x) = \varphi(x), \quad x \in \Gamma_-, \tag{2}$$

where $\Omega \subset \mathbb{R}^2$ is a regular domain with boundary $\Gamma = \partial\Omega$ and inflow boundary Γ_-.

(a) Compute the characteristics of (1).

(b) Assume $g(u) = u + 1$, $\varphi(x) = x_1^2$ and

$$\Omega = \{x \in \mathbb{R}^2 : 0 < x_1 < \infty \text{ and } 0 < x_2 < x_1\}.$$

Compute Γ_- and solve the differential equation (1) subject to the boundary condition (2).

(c) Now choose

$$\Omega = \{x \in \mathbb{R}^2 : |x| < 1\},$$

$g(u) = 0$ and φ arbitrarily. Compute Γ_- and solve the differential equation (1) subject to the boundary condition (2). Sketch the domain Ω together with Γ_- and the characteristics.

Exercise 4.2. (2 + 3 + 3 = 8 points) Let $\Omega = (a, b) \subset \mathbb{R}$ be an open interval.

(a) Suppose $\mu \in C_0^\infty(\Omega)$ with $\int_\Omega \mu \,\mathrm{d}x = 1$. For $\psi \in C_0^\infty(\Omega)$ we define $c = \int_\Omega \psi \,\mathrm{d}x$ and

$$\varphi(x) = \int_a^x \psi(y) - c\mu(y) \,\mathrm{d}y, \quad x \in \mathbb{R}.$$

Show that $\varphi \in C_0^\infty(\Omega)$.

(b) Show that $u \in H^1(\Omega)$ with $u' = 0$ a. e. implies $u = C$ a. e. for a constant $C \in \mathbb{R}$.
Hint: Construct test functions as in (a) and conclude with Exercise 1.3.

(c) Suppose $k \in \mathbb{N}$ and $u \in H^k(\Omega)$. Show that if $u^{(k)} = 0$ a. e., then there is a polynomial p of degree less or equal $k - 1$ such that $u = p$ a. e.

Exercise 4.3. (2 + 2 = 4 points) In this exercise we consider the three-dimensional wave equation

$$\partial_t^2 u - \Delta_x u = 0$$

for $u : \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}$. As initial conditions, we set

$$u(0,x) = \frac{\varphi(|x|)}{|x|} \quad \text{and} \quad \partial_t u(0,x) = 0$$

for $\varphi \in C_0^2(\mathbb{R}_+)$.

(a) Make the radially symmetric ansatz

$$u(t,x) = w(t,|x|) \quad \text{with } w : [0,\infty) \times \mathbb{R}_{\geq 0} \to \mathbb{R},$$

followed by the substitution $w(t,r) = z(t,r)/r$ and show that z satisfies

$$\begin{aligned} \partial_t^2 z - \partial_r^2 z &= 0 && \text{on } (0,\infty) \times \mathbb{R}_{>0}, \\ z(0,r) &= \varphi(r), && r \in \mathbb{R}_{\geq 0}, \\ \partial_t z(0,r) &= 0, && r \in \mathbb{R}_{\geq 0}. \end{aligned}$$

(b) Solve the above initial value problem and give the solution of the initial value problem for the three-dimensional wave equation.
Hint: Extend z and φ by odd reflection on $\mathbb{R}$, i.e. $\tilde{z}(t,r) = -z(t,-r)$ for $r < 0$.