

PDE and Boundary-Value Problems

Winter Term 2015/2016

Lecture 10

Saarland University

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Purpose of Lesson

- With the help of the Laplace transform, to illustrate an important concept known as **Duhamel's principle**.
- To show how the term u_x in the diffusion equation represents the phenomenon of convection. With the help of the Laplace transform, to solve the convection problem.

Heat Flow within a Rod with Temperature Fixed on the Boundaries

Quite often, it is important to find the temperature inside a medium due to **time-varying boundary conditions**. For example, consider an insulated rod with temperature specified as $f(t)$ on the right end.

Problem 10-1

$$\text{PDE:} \quad u_t = u_{xx}, \quad 0 < x < 1, \quad 0 < t < \infty$$

$$\text{BCs:} \quad \begin{cases} u(0, t) = 0 \\ u(1, t) = f(t) \end{cases}, \quad 0 < t < \infty$$

$$\text{IC:} \quad u(x, 0) = 0, \quad 0 \leq x \leq 1$$

Heat Flow within a Rod with Temperature Fixed on the Boundaries

We may think that the solution to problem 10-1 can be easily found once we know the solution to the simpler version (constant temperature on the boundaries)

Problem 10-2

$$\text{PDE: } w_t = w_{xx}, \quad 0 < x < 1, \quad 0 < t < \infty$$

$$\text{BCs: } \begin{cases} w(0, t) = 0 \\ w(1, t) = 1 \end{cases}, \quad 0 < t < \infty$$

$$\text{IC: } w(x, 0) = 0, \quad 0 \leq x \leq 1$$

Heat Flow within a Rod with Temperature Fixed on the Boundaries

Solving problems 10-1 and 10-2 at the same time we have

Easy Problem 10-2 (constant BCs)	Hard Problem 10-1 (time-varying BCs)
<div data-bbox="182 498 665 635" style="border: 1px solid black; padding: 5px; margin-bottom: 10px;"> Transform problem 10-2 by the Laplace transform </div> $\frac{d^2}{dx^2} W - sW(x) = 0$ $W(0) = 0$ $W(1) = 1/s$	<div data-bbox="709 498 1192 635" style="border: 1px solid black; padding: 5px; margin-bottom: 10px;"> Transform problem 10-1 by the Laplace transform </div> $\frac{d^2}{dx^2} U - sU(x) = 0$ $U(0) = 0$ $U(1) = F(s)$

Easy Problem 10-2 (cont.) (constant BCs)

Solve the ODE

$$W(x, s) = \frac{1}{s} \left[\frac{\sinh(x\sqrt{s})}{\sinh(\sqrt{s})} \right]$$

Find the inverse transform

$$w(x, t) = x + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{-(n\pi)^2 t} \sin(n\pi x)$$

Hard Problem 10-1 (cont.) (time-varying BCs)

Solve the ODE

$$U(x, s) = F(s) \left[\frac{\sinh(x\sqrt{s})}{\sinh(\sqrt{s})} \right]$$

Don't invert yet - first multiply and divide by s

$$U = F(s) \left\{ s \left[\frac{\sinh(x\sqrt{s})}{s \sinh(\sqrt{s})} \right] \right\}$$

Easy Problem 10-2 (cont.) (constant BCs)	Hard Problem 10-1 (cont.) (time-varying BCs)
	<div data-bbox="701 275 1211 477" style="border: 1px solid black; padding: 10px; margin-bottom: 20px;"> <p>Using the relationship</p> $\mathcal{L}[w_t] = sW - w(x, 0)$ <p>we get</p> </div> $U(x, s) = F(s) \cdot \mathcal{L}[w_t]$ <p style="text-align: center;">⇓</p> $\begin{aligned} u(x, t) &= \mathcal{L}^{-1} \{ F(s) \cdot \mathcal{L}[w_t] \} \\ &= \mathcal{L}^{-1} [F(s)] * w_t \\ &= f(t) * w_t(t) \end{aligned}$

In other words,

$$\begin{aligned}
 u(x, t) &= \int_0^t w_t(x, t - \tau) f(\tau) d\tau \\
 &= \int_0^t w(x, t - \tau) f'(\tau) d\tau + f(0) w(x, t).
 \end{aligned}
 \tag{10.1}$$

Remarks

- Equation (10.1) is known as **Duhamel's principle**.
- Often, even the easy problem (constant BCs) cannot be solved analytically. However, we can observe the solution **experimentally**.
- When we have this experimental data, we can solve the problem with arbitrary BCs using Duhamel's formula.

The Convection term u_x in the Diffusion Problems

- Consider now the problem of finding the **concentration** of a substance upwards from the surface of the earth where the substance both diffuses through the air and is **carried upward** (convected) by moving currents (moving with velocity V).
- **Diffusion** is mixing the substance through the air, while **convection** is the movement of the substance by means of the air (the movement of the medium).
- It is possible for the convection of the substance to contribute more of a movement in the substance than the diffusion itself.

Derivation of the Diffusion-Convection Equation

To derive this equation for a concentration $u(x, t)$ of a substance, we use two basic facts

1 Flux due to convection

The flux of the material (from left to right) across a point due to **convection** is given by $Vu(x, t)$, where V is the linear velocity of the medium (cm/sec) and $u(x, t)$ is the linear concentration (g/sm)

2 Flux due to diffusion

The flux of material (from left to the right) across a point due to **diffusion** is given by $-Du_x(x, t)$, where D is the diffusion coefficient

Derivation of the Diffusion-Convection Equation

If we substitute these two expressions into the **conservation equation** (see Lecture 4), we get the basic PDE

$$u_t = Du_{xx} - Vu_x$$

Solution of the convection equation

- First, we consider the problem that is pure convection (the diffusion term is zero).
- A typical problem would be dumping a substance into a clean river (moving with velocity V) and observing the concentration of the substance downstream.

Problem 10-3

$$\text{PDE: } u_t = -Vu_x, \quad 0 < x < \infty, \quad 0 < t < \infty$$

$$\text{BC: } u(0, t) = P, \quad \leftarrow \text{Constant input of the substance}$$

$$\text{IC: } u(x, 0) = 0, \quad \leftarrow \text{Initially a clean river}$$

Step 1. (Transformation)

- Using the Laplace transform we get an ODE in x

$$\text{ODE: } sU(x) = -V \frac{d}{dx} U, \quad 0 < x < \infty$$

$$\text{BC: } U(0) = \frac{P}{s},$$

Step 2. (Solving the problem for ODE)

$$U(x) = \frac{P}{s} e^{-(sx/V)}$$

Step 3. (Inverse transform)

$$u(x, t) = \mathcal{L}^{-1}[U] = PH(t - x/V) = \begin{cases} 0, & t < x/V \\ P, & t \geq x/V \end{cases}$$

Solution of the convection equation

- The situation becomes more interesting when the solute (pollutant) **diffuses** with the medium.

Problem 10-4

$$\text{PDE: } u_t = Du_{xx} - Vu_x, \quad -\infty < x < \infty$$

$$\text{IC: } u(x, 0) = 1 - H(x), \quad -\infty < x < \infty$$

- In problem 10-4, we have moved the boundary to $-\infty$ (we have an IVP).
- To solve problem 10-4, we could use the Laplace transform in t or the Fourier transform in x . Alternative approach is introducing a new coordinate ξ .

So, our goal is to transform the IVP 10-4 into a **new one** in the moving coordinate system, solve it, and then transform back to get the solution in terms of the original coordinates (x, t) .

Step 1. (Change of variables)

- In place of the old coordinates (x, t) , we introduce new ones (ξ, τ)

$$\xi = x - Vt$$

$$\tau = t$$

- To rewrite the PDE in terms of (ξ, τ) , we use the chain rule

$$u_t = u_\xi \xi_\tau + u_\tau \tau_t = -Vu_\xi + u_\tau$$

$$u_x = u_\xi \xi_x = u_\xi$$

$$u_{xx} = (u_\xi)_x = u_{\xi\xi} \xi_x = u_{\xi\xi}$$

- Substituting our computed u_t , u_x and u_{xx} into the PDE, we get the new IVP in terms of ξ and τ

Problem 10-5

$$\text{PDE:} \quad u_t = Du_{\xi\xi}, \quad -\infty < \xi < \infty$$

$$\text{IC:} \quad u(\xi, 0) = 1 - H(\xi), \quad -\infty < \xi < \infty$$

Step 2. (Solving the transformed problem)

- The problem 10-5 has already been solved in Lecture 8 by the Fourier transform and has the solution

$$u(\xi, \tau) = \frac{1}{2\sqrt{D\pi\tau}} \int_{-\infty}^{\infty} \phi(\beta) e^{-(\xi-\beta)^2/(4D\tau)} d\beta,$$

where $\phi(\beta)$ is the initial condition.

- Hence, in our case, we have

$$u(\xi, \tau) = \frac{1}{2\sqrt{D\pi\tau}} \int_{-\infty}^0 e^{-(\xi-\beta)^2/(4D\tau)} d\beta$$

Step 2. (Solving the transformed problem)

- By letting

$$\bar{\beta} = \frac{\xi - \beta}{2\sqrt{D\tau}} \quad d\bar{\beta} = \frac{-1}{2\sqrt{D\tau}} d\beta$$

we get

$$\begin{aligned}
 u(\xi, \tau) &= \frac{1}{2} \left[\frac{2}{\sqrt{\pi}} \int_{\frac{\xi}{2\sqrt{D\tau}}}^{\infty} e^{-\bar{\beta}^2} d\bar{\beta} \right] \\
 &= \begin{cases} \frac{1}{2} \left[1 + \operatorname{erf} \left(\frac{-\xi}{2\sqrt{D\tau}} \right) \right], & \xi < 0 \\ \frac{1}{2} \operatorname{erfc} \left(\frac{\xi}{2\sqrt{D\tau}} \right), & \xi \geq 0 \end{cases}
 \end{aligned} \tag{10.2}$$

Step 3. (Change to the original variables)

- Finally, to get the solution of problem 10-4 in terms of the coordinates x and t , we substitute

$$\xi = x - Vt$$

$$\tau = t$$

into equation (10.2) to get

$$u(x, t) = \begin{cases} \frac{1}{2} \left[1 + \operatorname{erf} \left(\frac{Vt - x}{2\sqrt{Dt}} \right) \right], & Vt > x \\ \frac{1}{2} \operatorname{erfc} \left(\frac{x - Vt}{2\sqrt{Dt}} \right), & Vt \leq x \end{cases}$$