PDE and Boundary-Value Problems
Winter Term 2015/2016

Lecture 12

Saarland University

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Purpose of Lesson

- To discuss some properties of solutions to the heat equations
- To introduce the one-dimensional wave equation and show how it describes the motion of a vibrating string.
- To show how the one-dimensional wave equation is derived as a result of Newton's equations of motion.

Properties of solutions to the heat equation

1. (Strong maximum principle)

Assume $u \in C^{2,1}(U_T) \cap C(\overline{U_T})$ solves the heat equation in the parabolic cylinder U_T.

(i) Then

$$\max_{\overline{U_T}} u = \max_{\partial' U_T} u.$$

(ii) Furthermore, if U is connected and there exists a point $(x_0, t_0) \in U_T$ such that

$$u(x_0, t_0) = \max_{\overline{U_T}} u,$$

then

$$u \text{ is a constant in } \overline{U_{T_0}}.$$

Remarks

- Assertion (i) is the maximum principle for the heat equation and (ii) is the strong maximum principle.
- Similar assertions are valid with „min“ replacing „max“.
- So if u attains its maximum (or minimum) at the interior point, then u is constant at all earlier times. The solution may change at times $t > t_0$, provided the boundary conditions alter after t_0.

Properties of solutions to the heat equation (cont.)

2. (Uniqueness on bounded domains)

Let $g \in C(\partial' U_T)$ and $f \in C(U_T)$. Then there exists at most one solution

$$u \in C^{2,1}(U_T) \cap C(\overline{U_T})$$

of the problem

$$\begin{cases} u_t - \Delta u = f & \text{in } U_T \\ u = g & \text{on } \partial' U_T. \end{cases}$$

Properties of solutions to the heat equation (cont.)

3. (Smoothness)

Suppose $u \in C^{2,1}(U_T)$ solves the heat equation in U_T. Then

$$u \in C^{\infty}(U_T).$$

Remark

- The regularity assertion is valid even if u attains nonsmooth boundary values on $\partial' U_T$.

Chapter 3. Hyperbolic-Type Problems

So far, we have been concerned with physical phenomenon described by parabolic equations. We will now begin to study the second major class pf PDEs, hyperbolic equations.

We start by studying the one-dimensional wave equation, which describes (among other things) the transverse vibrations of a string.

Vibrating-String Problem

Suppose we have the following simple experiment that we break into steps.

1. Consider the small vibrations of a string length L that is fastened at each end.

2. We assume the string is stretched tightly, made of a homogeneous material, unaffected by gravity, and that the vibrations take place in a plane.

The mathematical model of the vibrating-string problem

To mathematically describe the vibrations of the 1-dimensional string, we consider all the forces acting on a small section of the string.

Essentially, the wave equation is nothing more than Newton's equation of motion applied to the string (the change of momentum mu_{tt} of a small string segment is equal to the applied forces).

The most important forces are

1. Net force due to the tension of the string ($\alpha^2 u_{xx}$)

   The tension component has a net transverse force on the string segment of

$$\begin{aligned}\text{Tension component} &= T\sin(\theta_2) - T\sin(\theta_1)\\ &\approx T\left[u_x(x+\Delta x, t) - u_x(x,t)\right]\end{aligned}$$

2. External force $F(x,t)$

   An external force $F(x,t)$ may be applied along the string at any value of x and t.

3. Frictional force against the string $(-\beta u_t)$

   If the string is vibrating in a medium that offers a resistance to the string's velocity u_t, then this resistance force is $-\beta u_t$.

4. Restoring force $(-\gamma u)$

   This is a force that is directed opposite to the displacement of the string. If the displacement u is positive (above the x-axis), then the force is negative (downward).

If we now apply Newton's equation of motion

$$mu_{tt} = \text{applied forces to the segment } (x, x + \Delta x)$$

to the small segment of string, we have

$$\begin{aligned}\Delta x \rho u_{tt}(x,t) = T\left[u_x(x+\Delta x, t) - u_x(x,t)\right] + \Delta x F(x,t) \\ - \Delta x \beta u_t(x,t) - \Delta x \gamma u(x,t),\end{aligned}$$

where ρ is the density of the string.

By dividing each side of the equation by Δx and letting $\Delta x \to 0$, we have the equation

$$u_{tt} = \alpha^2 u_{xx} - \delta u_t - \kappa u + f(x,t),$$

where $\alpha^2 = \dfrac{T}{\rho}$, $\delta = \dfrac{\beta}{\rho}$, $\kappa = \dfrac{\gamma}{\rho}$, and $f(x,t) = \dfrac{F(x,t)}{\rho}$.

Intuitive Interpretation of the Wave Equation

- The expression u_{tt} represents the vertical acceleration of the string at a point x.
- Equation

$$u_{tt} = \alpha^2 u_{xx}$$

  can be interpreted as saying that the acceleration of each point of the string is due to the tension in the string and that the larger the concavity u_{xx}, the stronger the force.

Remarks

- If the vibrating string had a variable density $\rho(x)$, then the wave equation would be

$$u_{tt} = \frac{\partial}{\partial x}\left[\alpha^2(x) u_x\right].$$

In other words, the PDE would have variable coefficients.

Remarks (cont.)

- Since the wave equation $u_{tt} = \alpha^2 u_{xx}$ contains a second-order time derivative u_{tt}, it requires two initial conditions

$$u(x,0) = f(x) \qquad \text{(initial position of the string)}$$
$$u_t(x,0) = g(x) \qquad \text{(initial velocity of the string)}$$

  in order to uniquely define the solution for $t > 0$. This is in contrast to the heat equation, where only one IC was required.