PDE and Boundary-Value Problems
Winter Term 2015/2016

Lecture 13

Saarland University

14. Dezember 2015

Purpose of Lesson

- To find D'Alembert solution of the wave equation and interpretate it in terms of moving wave motion.
- To interpretate the D'Alembert solution in the xt-plane.

The D'Alembert Solution of the Wave Equation

- In the parabolic case we started solving problems when the space variable was bounded (by separation of variables) and then went on to solve the unbounded case (where $-\infty < x < \infty$) by the Fourier transform.
- In the hyperbolic case (wave problem), we will do the opposite.
- We start by solving the one-dimensional wave equation in free spece. We will use the method similar to the moving-coordinate method from diffusion-comvection equation.

Problem 13-1

To find the function $u(x,t)$ that satisfies

$$\text{PDE:} \qquad u_{tt} = c^2 u_{xx}, \qquad -\infty < x < \infty, \quad 0 < t < \infty$$

$$\text{ICs:} \quad \begin{cases} u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases} \qquad -\infty < x < \infty$$

We solve problem 13-1 by breaking it into several steps.

Step 1. (Replacing (x, t) by new canonical coordinates (ξ, η))

- We introduce two new space-time coordinates (ξ, η)

$$\xi = x + ct$$
$$\eta = x - ct$$

- In new variables our PDE takes the form

$$u_{\xi\eta} = 0. \tag{13.1}$$

Step 2. (Solving the transformed equation)

- We solve (13.1) by two straightforward integrations (first with respect to ξ and then with respect to η). The general solution of (13.1) is

$$u(\xi, \eta) = \phi(\eta) + \psi(\xi), \tag{13.2}$$

where $\phi(\eta)$ and $\psi(\xi)$ are arbitrary functions of η and ξ, respectively.

Step 3. (Transforming back to the original coordinates x and t)

- We substitute

$$\xi = x + ct$$
$$\eta = x - ct$$

into (13.2) to get

$$u(x,t) = \phi(x - ct) + \psi(x + ct). \tag{13.3}$$

Remark

(13.3) is physically represents the sum of any two moving waves, each moving in opposite directions with velocity c.

Step 4. (Substituting the general solution into the ICs)

- Substituting (13.3) into our ICs, we get

$$\begin{aligned} \phi(x) + \psi(x) &= f(x) \\ -c\phi'(x) + c\psi'(x) &= g(x) \end{aligned} \tag{13.4}$$

- Integrating the second equation of (13.4) from x_0 to x, we obtain

$$-c\phi(x) + c\psi(x) = \int\limits_{x_0}^{x} g(s)ds + K. \tag{13.5}$$

Step 4. (Substituting the general solution into the ICs (cont.))

- If we solve algebraically for $\phi(x)$ and $\psi(x)$ from the first equation of (13.4) and (13.5), we have

$$\phi(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int\limits_{x_0}^{x} g(s)ds - \frac{K}{2c}$$

$$\psi(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int\limits_{x_0}^{x} g(s)ds + \frac{K}{2c}$$

Step 4. (Substituting the general solution into the ICs (cont.))

- Hence, the solution to our problem 13-1 is

$$\boxed{u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int\limits_{x-ct}^{x+ct} g(s)ds}.$$

It is called the D'Alembert solution.

Examples of the D'Alembert Solution

1. Motion of an Initial Sine Wave

- Consider the initial conditions

$$u(x,0) = \sin(x)$$
$$u_t(x,0) = 0$$

The initial sine wave would have the solution

$$u(x,t) = \frac{1}{2}\left[\sin(x - ct) + \sin(x + ct)\right]$$

- This can be interpreted as dividing the initial shape $u(x,0) = \sin(x)$ into two equal parts

$$\frac{\sin(x)}{2} \quad \text{and} \quad \frac{\sin(x)}{2}$$

and then adding the two resultant waves as one moves to the left and the other to the right (each with velocity c).

Examples of the D'Alembert Solution (cont.)

2. Motion of a Simple Square Wave
   - In this case, if we start the initial conditions

$$u(x,0) = \begin{cases} 1, & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$
$$u_t(x,0) = 0$$

then the initial wave is decomposed into two half waves travelling in opposite direction.

Examples of the D'Alembert Solution (cont.)

3. Initial Velocity Given
   - Suppose now the initial position of the string is at equilibrium and we impose an initial velocity (as in piano string) of $\sin(x)$

$$u(x,0) = 0$$
$$u_t(x,0) = \sin(x)$$

   - Here, the solution would be

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \sin(s)ds$$
$$= \frac{1}{2c}\left[\cos(x+ct) - \cos(x-ct)\right]$$

which represents the sum of two moving cosine wave.

Remarks

- Note that a second-order PDE has two arbitrary functions in its general solution, whereas the general solution of a second-order ODE has two arbitrary constants. In other words, there are more solutions to a PDE than to an ODE.

- The general technique of changing coordinate systems in a PDE in order to find a simpler equation is common in PDE theory.

- The new coordinates (ξ, η) in problem 13-1 are known as canonical coordinates.

- The strategy of finding the general solution to a PDE and then substituting it into the boundary and initial conditions is not a common technique in solving PDEs.

The Space-Time Interpretation of D'Alembert's Solution

We present an interpretation of the D'Alembert solution

$$u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds$$

in the *xt*-plane looking at two specific cases.

Case 1. (Initial position given; initial velocity zero)

- Suppose the string has initial conditions

$$u(x,0) = f(x)$$
$$u_t(x,0) = 0$$

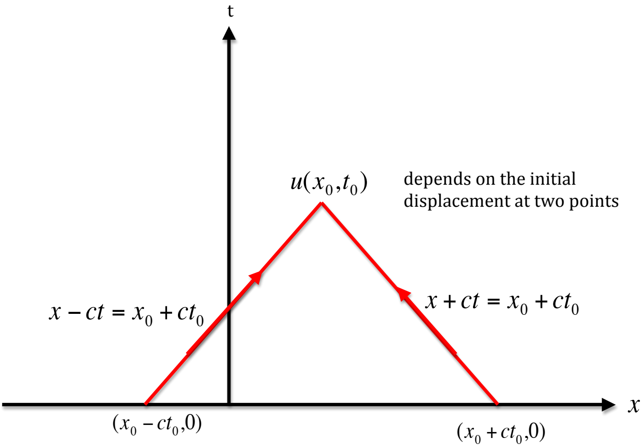
Here, the D'Alembert solution is

$$u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right].$$

- The solution u at a point (x_0, t_0) can be interpreted as being the average of the initial displacement $f(x)$ at the points $(x_0 - ct_0, 0)$ and $(x_0 + ct_0, 0)$ found by backtracking along the lines (characteristic curves)

$$x - ct = x_0 - ct_0$$
$$x + ct = x_0 + ct_0$$

Fig.13.1 Interpretation of $u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right]$ in the xt-plane



For example, using this interpretation, the IVP

Problem 13-2

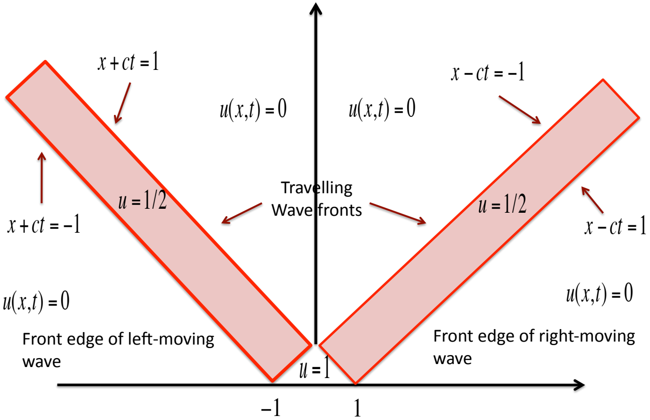
To find the function $u(x,t)$ that satisfies

$$\text{PDE:} \quad u_{tt} = c^2 u_{xx}, \qquad -\infty < x < \infty, \; 0 < t < \infty$$

$$\text{ICs:} \quad \begin{cases} u(x,0) = \begin{cases} 1, & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases} \\ u_t(x,0) = 0 \end{cases} \qquad -\infty < x < \infty$$

would give us the solution in the *xt*-plane shown in Fig. 13.2.

Fig. 13.2 Solution of problem 13-2 in the *xt*-plane



Case 2. (Initial displacement zero; velocity arbitrary)

- Consider now the ICs

$$u(x,0) = 0$$
$$u_t(x,0) = g(x)$$

Here, the D'Alembert solution is

$$u(x,t) = \frac{1}{2c} \int_{x-ct}^{x+ct} g(s)ds$$

- Hence, the solution u at (x_0, t_0) can be interpreted as integrating the initial velocity between $x_0 - ct_0$ and $x_0 + ct_0$ on the initial line $t = 0$.

Again, using this interpretation, the solution to the IVP

Problem 13-3

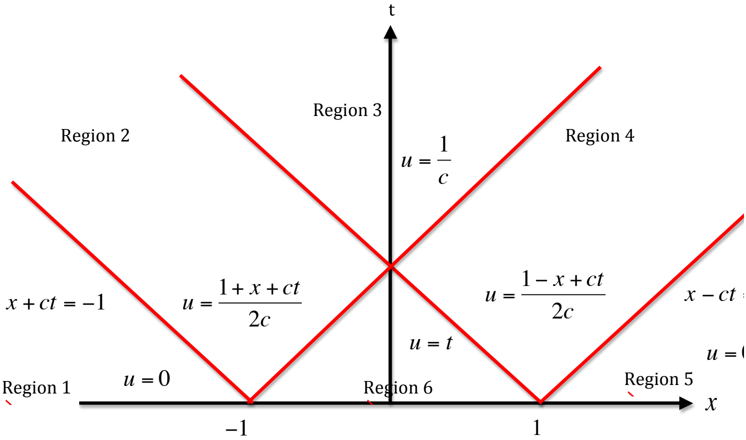
To find the function $u(x,t)$ that satisfies

$$\text{PDE:} \quad u_{tt} = c^2 u_{xx}, \qquad -\infty < x < \infty, \quad 0 < t < \infty$$

$$\text{ICs:} \quad \begin{cases} u(x,0) = 0 \\ u_t(x,0) = \begin{cases} 1, & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases} \end{cases} \qquad -\infty < x < \infty$$

has a solution in the *xt*-plane illustrated in Fig. 13.3.

Fig. 13.3 Solution of problem 13-3 in the *xt*-plane



Problem 13-3 corresponds to imposing an initial impulse (velocity $= 1$) on the string for $-1 < x < 1$ and watching the resulting wave motion (as in the piano string).

The solution is graphed at various values of times in Figures 13.4-13.4a.

Fig. 13.4 Solution of problem 13-3 for various values of time

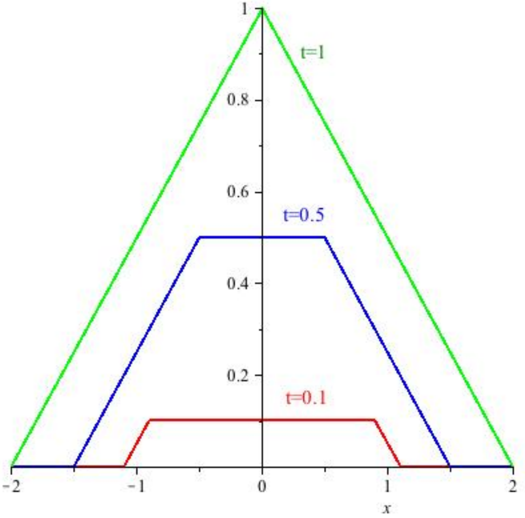
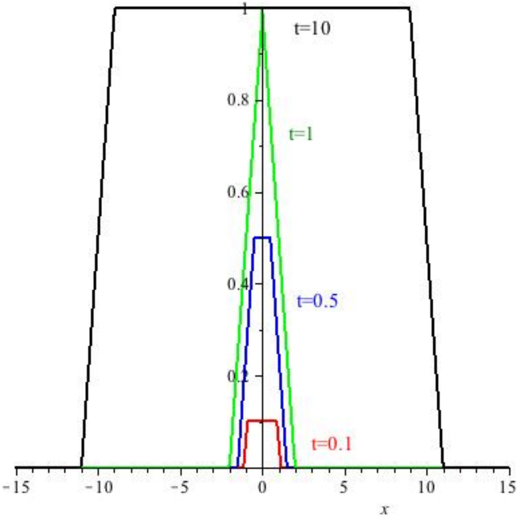


Fig. 13.4a Solution of problem 13-3 for various values of time



$$y = \frac{\ln\left(\frac{x}{m} - sa\right)}{r^2}$$

$$yr^2 = \ln\left(\frac{x}{m} - sa\right)$$

$$e^{yr^2} = \frac{x}{m} - sa$$

$$me^{yr^2} = x - sam$$

$$\boxed{me^{rry} = x - mas}$$