

PDE and Boundary-Value Problems

Winter Term 2015/2016

Lecture 14

Saarland University

4. Januar 2016

Purpose of Lesson

- To illustrate how the D'Alembert solution can be used to find the wave motion of a **semi-infinite-string** problem.
- To illustrate how the boundary conditions is generally associated with the wave equation.

Solution of the Semi-infinite String via the D'Alembert Formula

We will solve the IBVP for the semi-infinite string

Problem 14-1

To find the function $u(x, t)$ that satisfies

$$\text{PDE:} \quad u_{tt} = c^2 u_{xx}, \quad 0 < x < \infty, \quad 0 < t < \infty$$

$$\text{BC:} \quad u(0, t) = 0, \quad 0 < t < \infty$$

$$\text{ICs:} \quad \begin{cases} u(x, 0) = f(x) \\ u_t(x, 0) = g(x) \end{cases} \quad 0 < x < \infty$$

by **modifying** the D'Alembert formula. To find the solution of problem 14-1, we proceed in a manner similar to that used with the infinite string.

- We find the general solution to the PDE

$$u(x, t) = \phi(x - ct) + \psi(x + ct).$$

- Substituting this general solution into ICs we arrive at

$$\begin{aligned}\phi(x - ct) &= \frac{1}{2}f(x - ct) - \frac{1}{2c} \int_{x_0}^{x-ct} g(s)ds \\ \psi(x + ct) &= \frac{1}{2}f(x + ct) + \frac{1}{2c} \int_{x_0}^{x+ct} g(s)ds\end{aligned}\tag{14.1}$$

We now have a problem that we didn't encounter when dealing with the **infinite** string.

- Since we are looking for the solution $u(x, t)$ for $x > 0$ and $t > 0$, it is obvious that we must find

$$\phi(x - ct) \quad \forall \quad -\infty < x - ct < \infty$$

$$\psi(x + ct) \quad \forall \quad 0 < x + ct < \infty.$$

- Unfortunately, the first equation in (14.1) only gives us $\phi(x - ct)$ for $x - ct \geq 0$, since our initial data $f(x)$ and $g(x)$ are only known for **positive** arguments.

- As long as $x - ct \geq 0$, we have

$$\begin{aligned}u(x, t) &= \phi(x - ct) + \psi(x + ct) \\ &= \frac{1}{2} [f(x - ct) + f(x + ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) ds.\end{aligned}$$

- The question is, what to do when $x < ct$?

- When $x < ct$, we use our BC. Substituting the general solution u into the BC $u(0, t) = 0$ gives

$$\phi(-ct) = -\psi(ct)$$

- Hence, by functional substitution

$$\phi(x - ct) = -\psi(ct - x) = -\frac{1}{2}f(ct - x) - \frac{1}{2c} \int_{x_0}^{ct-x} g(s) ds$$

- Substituting this value of ϕ into the general solution gives

$$u(x, t) = \frac{1}{2} [f(x + ct) - f(ct - x)] + \frac{1}{2c} \int_{ct-x}^{x+ct} g(s) ds \quad 0 < x < ct.$$

- Combining the solutions for $x < ct$ and $x > ct$ we have our result

$$u(x, t) = \begin{cases} \frac{1}{2} [f(x - ct) + f(x + ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) ds, & x \geq ct \\ \frac{1}{2} [f(x + ct) - f(ct - x)] + \frac{1}{2c} \int_{ct-x}^{x+ct} g(s) ds, & x < ct \end{cases}$$

Remarks

- Solution of problem 14-1 would not be the same if the BC $u(0, t) = 0$ were changed. Solutions can **also** be found with other BCs, such as

$$u(0, t) = f(t) \quad \text{or} \quad u_x(0, t) = 0.$$

- The straight lines

$$x + ct = \text{constant}$$

$$x - ct = \text{constant}$$

are known as **characteristics**, and it is along these lines that disturbances are propagated. Characteristics are generally associated with hyperbolic equations.

Boundary Conditions Associated with the Wave Equation

- We have discussed the one-dimensional transverse vibrations of a string. A few other types of important vibrations are:
 - 1 Sound waves (longitudinal waves)
 - 2 Electromagnetic waves of light and electricity
 - 3 Vibrations in solids (longitudinal, transverse, and torsional)
 - 4 Probability waves in quantum mechanics
 - 5 Water waves (transverse waves)
 - 6 Vibrating string (transverse waves)
- We will discuss some of the various types of BCs that are associated with physical problems of this kind.

We will stick to **one-dimensional problems** where the BCs (linear ones) are generally grouped in to one of three kinds:

1. **Controlled end points** (first kind)

$$u(0, t) = g_1(t)$$

$$u(L, t) = g_2(t)$$

2. **Force given on the boundaries** (second kind)

$$u_x(0, t) = g_1(t)$$

$$u_x(L, t) = g_2(t)$$

3. **Elastic attachment on the boundaries** (third kind)

$$u_x(0, t) - \gamma_1 u(0, t) = g_1(t)$$

$$u_x(L, t) - \gamma_2 u(L, t) = g_2(t)$$

1. Controlled End Points

We are now involved with problems like

Problem 14-2

To find the function $u(x, t)$ that satisfies

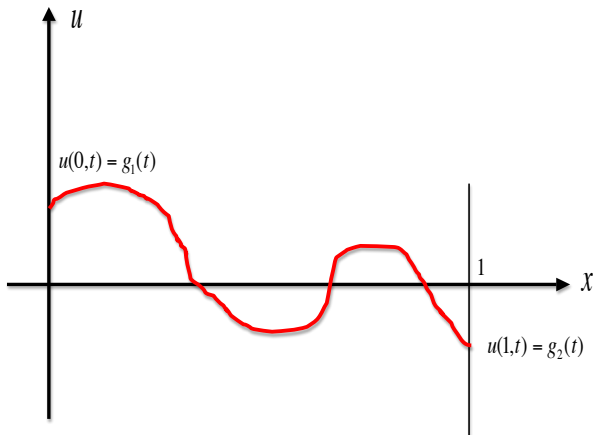
$$\text{PDE: } u_{tt} = c^2 u_{xx}, \quad 0 < x < 1, \quad 0 < t < \infty$$

$$\text{BCs: } \begin{cases} u(0, t) = g_1(t) \\ u(1, t) = g_2(t) \end{cases} \quad 0 < t < \infty$$

$$\text{ICs: } \begin{cases} u(x, 0) = f(x) \\ u_t(x, 0) = g(x) \end{cases} \quad 0 \leq x \leq 1$$

where we **control** the end points so that they move in a given manner.

Fig. 14.1 Controlling the ends of a vibrating string

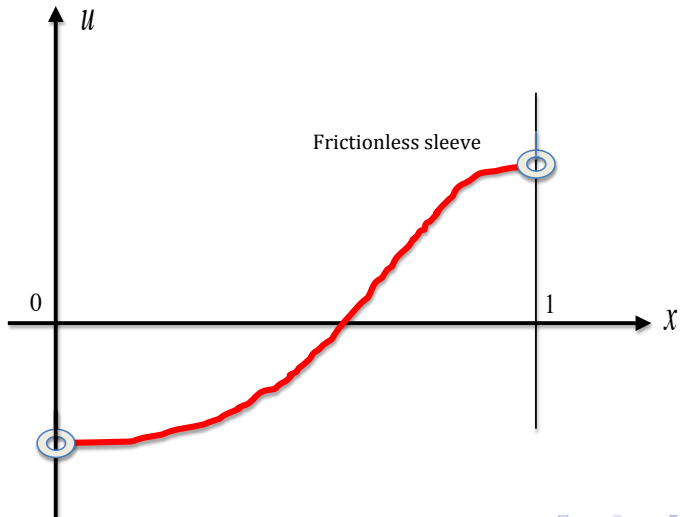


2. Force Given on the Boundaries

Inasmuch as the **vertical forces** on the string at the left and right ends are given by $Tu_x(0, t)$ and $Tu_x(L, t)$, respectively, by allowing the ends of the string to slide vertically on frictionless, the BCs become

$$\begin{aligned}u_x(0, t) &= 0 \\u_x(L, t) &= 0\end{aligned}\tag{14.2}$$

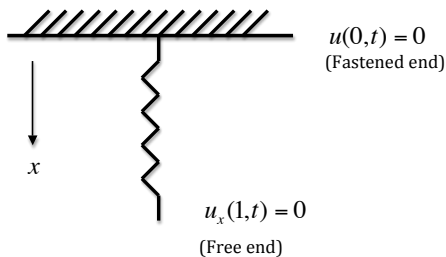
Fig. 14.2 Free BC on the string



BCs similar to (14.2) are presented in the following two examples:

a) Free end of a longitudinally vibrating spring

Consider a vibrating spring with the bottom end unfastened



b) Forced end of a vibrating spring

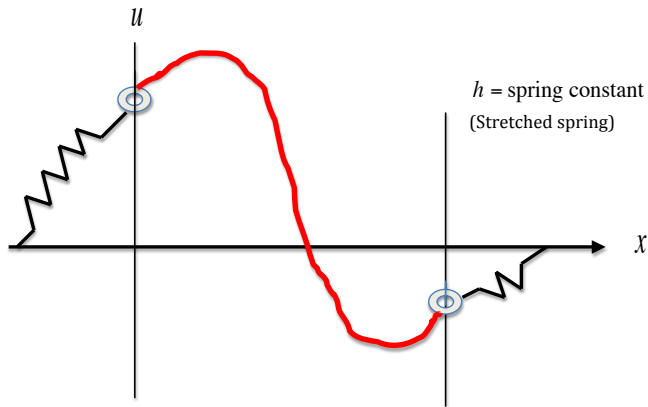
- If a force of $v(t)$ dynes is applied at the end $x = 1$ (a positive force is measured downward), then the BC would be

$$u_x(1, t) = \frac{1}{k} v(t) \quad (k \text{ is Young's modulus})$$

- In the case of a forced BC, the ends of the string (or spring) are not **required** to maintain a given position, but the force that's applied tends to move the boundaries in the given direction.

3. Elastic Attachment on the Boundaries

Consider finally a violin string whose ends are attached to an elastic arrangement



3. Elastic Attachment on the Boundaries

The spring attachments at each end give rise to vertical forces proportional to the displacements

$$\text{Displacement at the left end} = u(0, t)$$

$$\text{Displacement at the right end} = u(L, t)$$

Setting the vertical tensions of the spring at the two ends

$$\text{Upward tension at the left end} = Tu_x(0, t) \quad (T = \text{string tension})$$

$$\text{Upward tension at the right end} = -Tu_x(L, t)$$

equal to these displacements (multiplied by the spring constant h) gives us our desired BCs:

3. Elastic Attachment on the Boundaries

$$u_x(0, t) = \frac{h}{T} u(0, t)$$

$$u_x(L, t) = -\frac{h}{T} u(L, t)$$

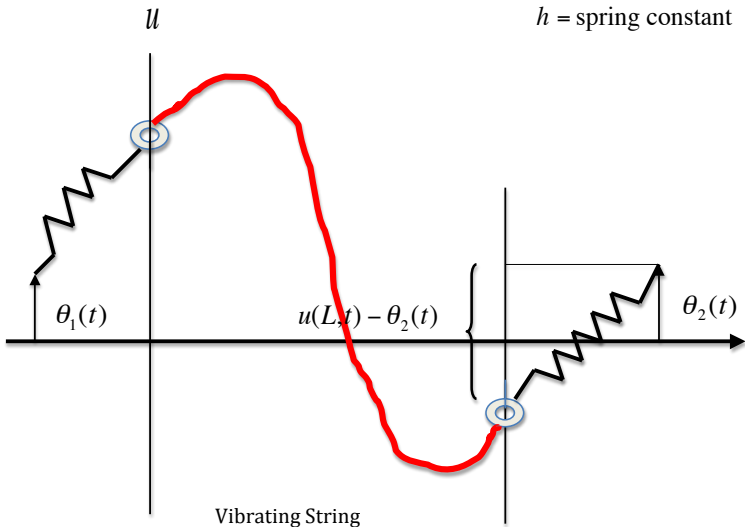
Remark

Note that $u(0, t)$ positive means that $u_x(0, t)$ is positive, while if $u(L, t)$ is positive, then $u_x(L, t)$ is negative.

If the two spring attachments are displaced according to the functions $\theta_1(t)$ and $\theta_2(t)$, we would have the **nonhomogeneous** BCs

$$u_x(0, t) = \frac{h}{T} [u(0, t) - \theta_1(t)]$$

$$u_x(L, t) = -\frac{h}{T} [u(L, t) - \theta_2(t)]$$



Remarks

- Another BC not discussed today occurs when the vibrating string experiences a force at the ends proportional to the string velocity (and in the opposite direction). Here, we have the BC (at the left end)

$$T u_x(0, t) = -\beta u_t(0, t)$$

- A nonlinear elastic attachment at the left end of the string would be

$$T u_x(0, t) = \phi [u(0, t)]$$

where $\phi(u)$ is an arbitrary function of u ; for example

$$T u_x(0, t) = -h u^3(0, t)$$

Remarks (cont.)

- If a mass m is attached to the lower end of a longitudinally vibrating string, the BC would be

$$mu_{tt}(L, t) = -ku_x(L, t) + mg$$