

# PDE and Boundary-Value Problems

## Winter Term 2015/2016

### Lecture 15

Saarland University

8. Januar 2016

## Purpose of Lesson

- To show how transverse vibrations of a finite string can be found by the standard technique of separation of variables and to show how the solution  $u(x, t)$  can be interpreted as the infinite sum

$$u(x, t) = \sum_{n=1}^{\infty} X_n(x) T_n(t)$$

of **simple vibrations where the shape**  $X_n(x)$  of these fundamental vibrations are solutions (eigenfunctions) of a certain Sturm-Liouville BVP.

## Purpose of Lesson (cont.)

- To illustrate how higher-order PDEs come about in the study of vibrating-beam problems.
- To solve the problem of a vibrating beam with simply supported ends by separation of variables.
- To compare the vibrations of the beam with the vibrations of the violin string.

# The Finite Vibrating String (Standing Waves)

- So far, we have studied the wave equation  $u_{tt} = c^2 u_{xx}$  for the unbounded domain  $-\infty < x < \infty$  and have found (D'Alembert's solution) solutions to be certain **travelling** waves (moving in opposite directions).
- When we study the same wave equation in a **bounded region** of space  $0 < x < L$ , we find that the waves no longer **appear** to be moving due to their repeated interaction with the boundaries and, in fact, often appear to be what are known as **standing waves**.

Consider what happens when a guitar string (fixed at both ends  $x = 0, L$ ) described by the simple hyperbolic IBVP

### Problem 15-1

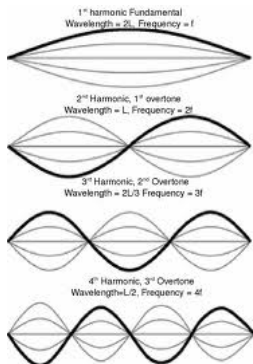
To find the function  $u(x, t)$  that satisfies

$$\text{PDE:} \quad u_{tt} = c^2 u_{xx}, \quad 0 < x < L, \quad 0 < t < \infty$$

$$\text{BCs:} \quad \begin{cases} u(0, t) = 0 \\ u(L, t) = 0 \end{cases} \quad 0 < t < \infty$$

$$\text{ICs:} \quad \begin{cases} u(x, 0) = f(x) \\ u_t(x, 0) = g(x) \end{cases} \quad 0 \leq x \leq L$$

What happens is that the travelling-wave solution to the PDE and IC keeps reflecting from the boundaries in such a way that the wave motion does not appear to be moving, but, in fact, appears to be vibrating in one position.



If we **knew** the shapes  $X_n(x)$  of these standing waves and how each one of them vibrated  $T_n(t)$ , then all we would have to do to find the solution of the vibrating guitar string is sum the simple vibrations  $X_n(x)T_n(t)$

$$u(x, t) = \sum_{n=1}^{\infty} c_n X_n(x) T_n(t)$$

in such a way (find the coefficients  $c_n$ ) that the sum agrees with the ICs

$$u(x, 0) = f(x)$$

$$u_t(x, 0) = g(x)$$

# Separation-of-Variables Solution to the Finite Vibrating String

We solve problem 15-1 by breaking it into several steps:

## Step 1. (Separation of Variables)

- We start by seeking solutions to the PDE of the form

$$u(x, t) = X(x)T(t)$$

- Substituting this expression into the wave equation and separating variables gives us the two ODEs

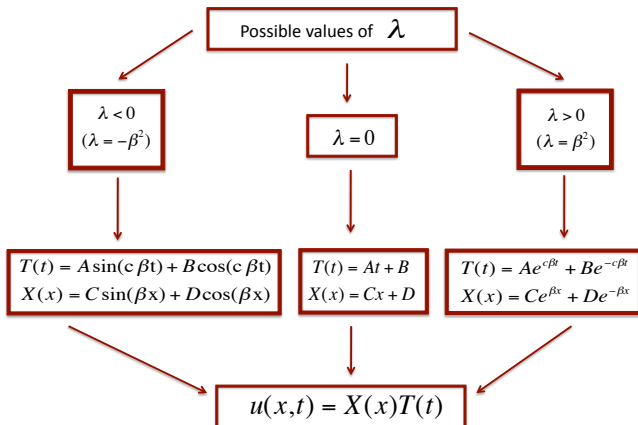
$$T'' - c^2\lambda T = 0$$

$$X'' - \lambda X = 0$$

where the constant  $\lambda$  can now be any number  $-\infty < \lambda < \infty$ .



## Step 2. (Solving ODEs)



### Step 3. (Substituting into BCs)

- The idea now is to **prune away** all those standing waves that either are unbounded as  $t \rightarrow \infty$  or else yield only the zero solution when substituted into the BCs.
- Only negative values of  $\lambda$  give nonzero and bounded solutions. Hence,

$$u(x, t) = [C \sin(\beta x) + D \cos(\beta x)] [A \sin(c\beta t) + B \cos(c\beta t)]$$

- Substitution expression of  $u$  into BCs gives

$$D = 0$$
$$\beta_n = \frac{\pi n}{L}, \quad n = 0, 1, 2, \dots$$

### Step 3. (Substituting into BCs)

- We have found a **sequence** of simple vibrations (which we subscript with  $n$ )

$$\begin{aligned}u_n(x, t) &= \sin\left(\frac{n\pi x}{L}\right) \left[ a_n \sin\left(\frac{n\pi ct}{L}\right) + b_n \cos\left(\frac{n\pi ct}{L}\right) \right] \\ &= R_n \sin\left(\frac{n\pi x}{L}\right) \cos\left[\frac{n\pi c(t - \delta_n)}{L}\right],\end{aligned}$$

where the constants  $a_n$ ,  $b_n$ ,  $R_n$  and  $\delta_n$  are arbitrary. These simple vibrations satisfy the wave equations and the BCs.

## Step 4. (Substituting into ICs)

- Since any sum of these vibrations is also a solution to the PDE and BCs (since PDE and BCs are linear and homogeneous), we add them together in such a way that the resulting sum **also** agrees with the ICs.
- Substituting the sum

$$u(x, t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \left[ a_n \sin\left(\frac{n\pi ct}{L}\right) + b_n \cos\left(\frac{n\pi ct}{L}\right) \right]$$

into the ICs gives the two equations

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) = f(x)$$

$$\sum_{n=1}^{\infty} a_n \left(\frac{n\pi c}{L}\right) \sin\left(\frac{n\pi x}{L}\right) = g(x)$$

## Step 4. (Substituting into ICs)

- Using the orthogonality condition

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx = \begin{cases} 0, & m \neq n \\ \frac{L}{2}, & m = n \end{cases}$$

we can find the coefficients  $a_n$  and  $b_n$

$$\begin{aligned} a_n &= \frac{2}{n\pi c} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx \\ b_n &= \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx \end{aligned} \tag{15.1}$$

## Step 4. (Substituting into ICs)

- The solution is

$$u(x, t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \left[ a_n \sin\left(\frac{n\pi ct}{L}\right) + b_n \cos\left(\frac{n\pi ct}{L}\right) \right],$$

where the coefficients  $a_n$  and  $b_n$  are given by (15.1).

## Remarks

- If the initial **velocity** of the string is zero, then the solution takes the form

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi ct}{L}\right)$$

and has the following interpretation. Suppose we break the initial string position into simple sine components

$$u(x, 0) = f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

and let each sine term vibrate on its own according to

$$u_n(x, t) = b_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi ct}{L}\right)$$

## Remarks (cont.)

- If we now add each individual vibration of the type (this is a fundamental vibration)

$$u_n(x, t) = b_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi ct}{L}\right),$$

we will get the solution of our problem.

- The  $n$ -th term in the solution

$$\sin\left(\frac{n\pi x}{L}\right) \left[ a_n \sin\left(\frac{n\pi ct}{L}\right) + b_n \cos\left(\frac{n\pi ct}{L}\right) \right]$$

is called  **$n$ -th mode of vibration** or the  **$n$ -th harmonic**.



## Remarks (cont.)

- By using a trigonometric identity, we can write this harmonic as

$$R_n \sin\left(\frac{n\pi x}{L}\right) \cos\left[\frac{n\pi c(t - \delta_n)}{L}\right],$$

where  $R_n$  and  $\delta_n$  are the new arbitrary constants (amplitude and phase angle). This new form of the  $n$ -th mode is more useful for analyzing the vibrations.

- The frequency  $\omega_n$  (rad / sec) of the  $n$ -th mode is

$$\omega_n = \frac{n\pi c}{L} = \frac{n\pi}{L} \sqrt{\frac{T}{\rho}}$$

( $T$ ,  $\rho$  are tension and density of the string, respectively).

## Remarks (cont.)

- The frequency  $\omega_n$  is  $n$  times the fundamental frequency  $\omega_1$ .

$$\omega_n = n \cdot \omega_1$$

- The property that all sound frequencies are multiples of a basic one is not shared by all types of vibrations.

This has something to do with the pleasing sound of a violin or guitar string in contrast to a drumhead, where the higher-order frequencies are not multiple frequencies of the fundamental one.

# The Vibrating Beam (Fourth-Order PDE)

- The major difference between the transverse vibrations of a violin string and the transverse vibrations of a thin beam is that the beam offers **resistance to bending**.
- The resistance is responsible for changing the wave equation to the fourth-order beam equation

$$u_{tt} = -\alpha^2 u_{xxxx},$$

where

$$\alpha^2 = K/\rho$$

$K$  = rigidity constant

$\rho$  = linear density of the beam

# The Simply Supported Beam

- Consider the small vibrations of a thin beam whose ends are simply fastened to two foundations.
- By „simply fastened“, we mean that the ends of the beam are held stationary, but the slopes at the end points can move ( the beam is held by a pin-type arrangement).
- It seems clear that the BCs at the ends of the beam should be

$$u(0, t) = 0$$

$$u(1, t) = 0$$

but what **isn't** so obvious is that the two BCs

$$u_{xx}(0, t) = 0$$

$$u_{xx}(1, t) = 0$$

also hold at the two ends.

# The Simply Supported Beam

- Hence, the vibrating beam can be described by the following IVBP ( $\alpha$  is set equal to one for simplicity)

## Problem 15-2

To find the function  $u(x, t)$  that satisfies

$$\text{PDE: } u_{tt} = -u_{xxxx}, \quad 0 < x < 1, \quad 0 < t < \infty$$

$$\text{BCs: } \begin{cases} u(0, t) = 0 \\ u_{xx}(0, t) = 0 \\ u(1, t) = 0 \\ u_{xx}(1, t) = 0 \end{cases} \quad 0 < t < \infty$$

$$\text{ICs: } \begin{cases} u(x, 0) = f(x) \\ u_t(x, 0) = g(x) \end{cases} \quad 0 \leq x \leq 1$$

## Step 1. (Separation of Variables)

To solve problem 15-2, we again use the separation of variables method and look for arbitrary periodic solutions; that is, vibrations of the form

$$u(x, t) = X(x)T(t).$$

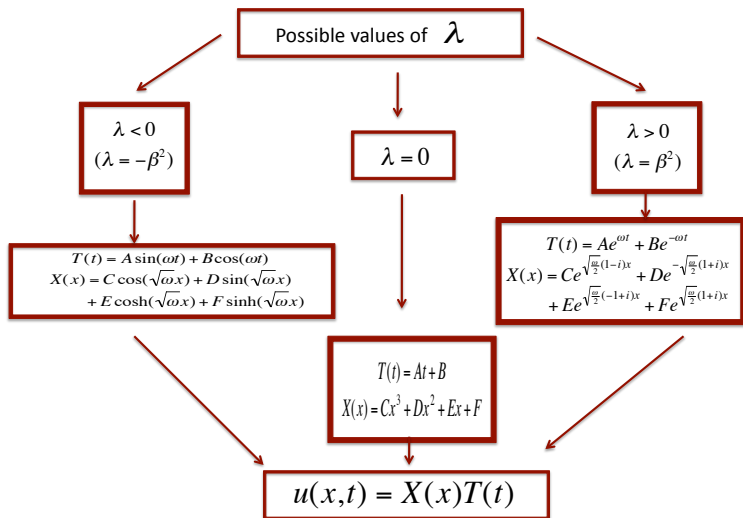
Substituting this expression into the wave equation and separating variables gives us the two ODEs

$$T'' - \lambda T = 0$$

$$X^{(iv)} + \lambda X = 0$$

where the constant  $\lambda$  can be any number  $-\infty < \lambda < \infty$ .

## Step 2. (Solving the ODEs)



### Step 3. (Substituting into BCs)

- Substituting into BCs immediately provides that for  $\lambda = 0$  and for  $\lambda = \omega^2 > 0$  we have

$$X(x) \equiv 0.$$

- So, the separation constant  $\lambda$  has been chosen to be **negative**, i.e.,  $\lambda = -\omega^2$ . Consequently,

$$X(x) = C \cos(\sqrt{\omega}x) + D \sin(\sqrt{\omega}x) \\ + E \cosh(\sqrt{\omega}x) + F \sinh(\sqrt{\omega}x)$$



## Step 3. (Substituting into BCs)

- Calculation of  $X''(x)$  gives

$$X''(x) = -\omega C \cos(\sqrt{\omega}x) - \omega D \sin(\sqrt{\omega}x) \\ + \omega E \cosh(\sqrt{\omega}x) + \omega F \sinh(\sqrt{\omega}x)$$

- Further, substitution of the expression for  $u$  into the BCs provides

$$u(0, t) = T(t) [C + E] = 0 \\ u_{xx}(0, t) = T(t) [-\omega C + \omega E] = 0 \quad \Rightarrow \quad \boxed{C = E = 0}.$$

$$u(1, t) = T(t) [D \sin(\sqrt{\omega}) + F \sinh(\sqrt{\omega})] = 0 \\ u_{xx}(1, t) = T(t) [-\omega D \sin(\sqrt{\omega}) + \omega F \sinh(\sqrt{\omega})] = 0$$

$$\Rightarrow \quad \boxed{\begin{aligned} F &= 0 \\ \sin(\sqrt{\omega}) &= 0 \end{aligned}}$$

## Step 2. (Substituting into BCs)

- Substituting the expression for  $u$  into the BCs, giving

$$C = E = F = 0$$

$$\omega = (\pi n)^2 \quad n = 1, 2, \dots$$

- Therefore, the **fundamental solutions**  $u_n$  (solutions of the PDE and BCs) are

$$\begin{aligned} u_n(x, t) &= X_n(x) T_n(t) \\ &= \left[ a_n \sin(\pi n)^2 t + b_n \cos(\pi n)^2 t \right] \sin(\pi n x) \end{aligned}$$

- Since the PDE and BCs are linear and homogeneous, we can conclude that the sum

$$u(x, t) = \sum_{n=1}^{\infty} \left[ a_n \sin(\pi n)^2 t + b_n \cos(\pi n)^2 t \right] \sin(\pi n x)$$

### Step 3. (Substituting into ICs)

- Substituting the expression for  $u$  into the ICs gives us

$$u(x, 0) = f(x) = \sum_{n=1}^{\infty} b_n \sin(\pi n x)$$

$$u_t(x, 0) = g(x) = \sum_{n=1}^{\infty} (\pi n)^2 a_n \sin(\pi n x)$$

- Using the fact that the family  $\{\sin(\pi n x)\}$  is orthogonal on the interval  $[0, 1]$  we arrive at

$$a_n = \frac{2}{(\pi n)^2} \int_0^1 g(x) \sin(\pi n x) dx$$

$$b_n = 2 \int_0^1 f(x) \sin(\pi n x) dx$$

## Remarks.

- Beams are generally fastened in one of three ways
  - 1 Free (unfastened)
  - 2 Simply fastened
  - 3 Rigidly fastened
- Another important vibrating-beam problem is the **cantilever-beam problem**. The solution to this vibrating beam is not the usual sum of products of sines and cosines, but due to the nonstandard BCs,

$$\begin{array}{ll}
 u(0, t) = 0 & u_{xx}(1, t) = 0 \\
 u_x(0, t) = 0 & u_{xxx}(1, t) = 0
 \end{array}$$

we arrive at the more complicated solution.

## Remarks (cont.)

- The solution of the cantilever-beam problem has the form

$$u(x, t) = \sum_{n=1}^{\infty} X_n(x) [a_n \sin(\omega_n t) + b_n \cos(\omega_n t)],$$

where the eigenfunctions (basic shapes of vibrations) are given by linear combinations of sines, cosines, hyperbolic sines and hyperbolic cosines.