PDE and Boundary-Value Problems
Winter Term 2015/2016

Lecture 19

Saarland University

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Purpose of Lesson

- To discuss an alternative integral-form of a solution (Poisson integral formula) to the interior Dirichlet problem.
- To solve the Dirichlet problem between two circles (annulus).
- To discuss briefly the solution to the exterior Dirichlet problem for the circle.
- To find particular solutions of the Laplace equation in spherical coordinates. To solve the interior and exterior Dirichlet problems for the Laplace equation in 3D.

Observations on the Dirichlet Solution:

- The interpretation of our solution (18.2) is that we should expand boundary function $g(\theta)$ as a Fourier series

  $$g(\theta) = \sum_{n=0}^{\infty} \left[a_n \cos(n\theta) + b_n \sin(n\theta) \right]$$

  and then solve the problem for each sine and cosine in the series.

  Since each of these terms will give rise to solutions $r^n \sin(n\theta)$ and $r^n \cos(n\theta)$, we can then say (by superposition) that

  $$u(r,\theta) = \sum_{n=0}^{\infty} r^n \left[a_n \cos(n\theta) + b_n \sin(n\theta) \right].$$

- The solution of

$$\text{PDE:} \quad \Delta u = 0, \qquad 0 < r < 1$$

$$\text{BC:} \quad u(1, \theta) = 1 + \sin\theta + \tfrac{1}{2}\sin(3\theta) + \cos(4\theta), \quad 0 \leqslant \theta < 2\pi.$$

would be

$$u(r, \theta) = 1 + r\sin\theta + \frac{r^3}{2}\sin(3\theta) + r^4\cos(4\theta).$$

Here, the $g(\theta)$ is already in the form of a Fourier series, with

$a_0 = 1$	$b_1 = 1$
$a_4 = 1$	$b_3 = 0.5$
All other a_n's $= 0$	All other b_n's $= 0$

and so we don't have to to use the formulas for a_n and b_n.

- If the radius of the circle were arbitrary (say R), then the solution would be

$$u(r,\theta) = \sum_{n=0}^{\infty} \left(\frac{r}{R}\right)^n \left[a_n \cos(n\theta) + b_n \sin(n\theta)\right].$$

- Note that the constant term a_0 in solution (18.2) represents the average of g

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} g(\alpha)\, d\alpha.$$

- We start with the separation of variables solution

$$u(r,\theta) = \sum_{n=0}^{\infty} \left(\frac{r}{R}\right)^n [a_n \cos(n\theta) + b_n \sin(n\theta)]$$

(we now take an arbitrary radius for the circle) and substitute the coefficients a_n and b_n.

- After a few manipulations involving algebra, calculus, and trigonometry, we have

$$u(r,\theta) = \frac{1}{2\pi}\int\limits_0^{2\pi} g(\theta)d\theta$$

$$+\frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n \int\limits_0^{2\pi} g(\alpha)\left[\cos(n\alpha)\cos(n\theta)+\sin(n\alpha)\sin(n\theta)\right]d\alpha$$

$$= \frac{1}{2\pi}\int\limits_0^{2\pi}\left\{1+2\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n\cos\left[n(\theta-\alpha)\right]\right\}g(\alpha)d\alpha$$

$$= \frac{1}{2\pi}\int\limits_0^{2\pi}\left\{1+\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n\left[e^{in(\theta-\alpha)}+e^{-in(\theta-\alpha)}\right]\right\}g(\alpha)d\alpha$$

$$= \frac{1}{2\pi}\int\limits_0^{2\pi}\left\{1+\frac{re^{i(\theta-\alpha)}}{R-re^{i(\theta-\alpha)}}+\frac{re^{-i(\theta-\alpha)}}{R-re^{-i(\theta-\alpha)}}\right\}g(\alpha)d\alpha$$

$$
\begin{aligned}
u(r,\theta) &= \frac{1}{2\pi}\int_0^{2\pi}\left\{1+\frac{re^{i(\theta-\alpha)}}{R-re^{i(\theta-\alpha)}}+\frac{re^{-i(\theta-\alpha)}}{R-re^{-i(\theta-\alpha)}}\right\}g(\alpha)d\alpha \\
&= \frac{1}{2\pi}\int_0^{2\pi}\left[\frac{R^2-r^2}{R^2-2rR\cos(\theta-\alpha)+r^2}\right]g(\alpha)d\alpha
\end{aligned}
\tag{19.1}
$$

It is the Poisson Integral Formula. So what we have is an alternative form for the solution to the interior Dirichlet problem.

Remarks

- We can interpret the Poisson integral solution (19.1) as finding the potential u at (r, θ) as a weighted average of the boundary potentials $g(\theta)$ weighted by the Poisson kernel

$$\text{Poisson kernel} = \frac{R^2 - r^2}{R^2 - 2rR\cos(\theta - \alpha) + r^2}.$$

- For boundary values $g(\alpha)$ close to (r, θ), the Poisson kernel gets large, since the denominator of the Poisson kernel is the square of the distance from (r, θ) to (R, α).

Remarks (cont.)

- Unfortunately, if (r, θ) is extremely close to the boundary $r = R$, then the Poisson kernel gets very large for those values of α that are closest to (r, θ). For this reason, when (r, θ) is close to the boundary, the series solution works better for evaluating the numerical value of the solution.

- If we evaluate the potential at the center of the circle by the Poisson integral, we find

$$u(0,0) = \frac{1}{2\pi} \int_0^{2\pi} g(\alpha) d\alpha.$$

In other words, the potential at the center of the circle is the average of the boundary potentials.

Remarks (cont.)

- We can always solve the BVP (nonhomogeneous PDE)

$$\text{PDE:} \quad \Delta u = f, \quad \text{Inside } D$$

$$\text{BC:} \quad u = 0, \quad \text{On the boundary of } D$$

by

1. Finding any solution V of $\Delta V = f$ (A particular solution).
2. Solving the new BVP

$$\text{PDE:} \quad \Delta W = 0, \quad \text{Inside } D$$

$$\text{BC:} \quad W = V, \quad \text{On the boundary of } D$$

3. Observing that $u = V - W$ is our desired solution.

In other words, we can transfer the nonhomogeneity from the PDE to BC.

Remarks (cont.)

- We can solve the BVP (nonhomogeneous BC)

$$\text{PDE:} \quad \Delta u = 0, \quad \text{Inside } D$$

$$\text{BC:} \quad u = f, \quad \text{On the boundary of } D$$

by

1. Finding any solution V that satisfies $V = f$ on the boundary of D.
2. Solving the new BVP

$$\text{PDE:} \quad \Delta W = \Delta V, \quad \text{Inside } D$$

$$\text{BC:} \quad W = 0, \quad \text{On the boundary of } D$$

3. Observing that $u = V - W$ is the solution to our problem.

In other words, we can transfer the nonhomogeneity from the BC to the PDE.

The Dirichlet Problem in an Annulus

Problem 19-1

To find the function $u(r,\theta)$ that satisfies

$$\text{PDE:}\quad u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad R_1 < r < R_2$$

$$\text{BCs:}\quad \begin{cases} u(R_1,\theta) = g_1(\theta), \\ u(R_2,\theta) = g_2(\theta) \end{cases} \qquad 0 \leqslant \theta < 2\pi.$$

Step 1. (Separation of Variables)

- Substituting $u(r,\theta) = R(r)\Theta(\theta)$ into the Laplace equation and arguing similar to the interior Dirichlet problem we arrive at our general solution

$$\boxed{\begin{aligned} u(r,\theta) = {} & a_0 + b_0 \ln r \\ & + \sum_{n=1}^{\infty} (a_n r^n + b_n r^{-n}) \cos(n\theta) \\ & + \sum_{n=1}^{\infty} (c_n r^n + d_n r^{-n}) \sin(n\theta) \end{aligned}} \tag{19.2}$$

Step 2. (Substituting into BCs)

- Substituting the solution (19.2) into the BCs and integrating gives the following equations:

$$\begin{cases} a_0 + b_0 \ln R_1 = \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} g_1(s)ds \\ a_0 + b_0 \ln R_2 = \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} g_2(s)ds \end{cases} \qquad \text{(Solve for } a_0, b_0\text{)}$$

Step 2. (cont.)

$$\begin{cases} a_n R_1^n + b_n R_1^{-n} = \dfrac{1}{\pi} \displaystyle\int_0^{2\pi} g_1(s) \cos(ns) ds \\ a_n R_2^n + b_n R_2^{-n} = \dfrac{1}{\pi} \displaystyle\int_0^{2\pi} g_2(s) \cos(ns) ds \end{cases}$$

(Solve for a_n, b_n)

$$\begin{cases} c_n R_1^n + d_n R_1^{-n} = \dfrac{1}{\pi} \displaystyle\int_0^{2\pi} g_1(s) \sin(ns) ds \\ c_n R_2^n + d_n R_2^{-n} = \dfrac{1}{\pi} \displaystyle\int_0^{2\pi} g_2(s) \sin(ns) ds \end{cases}$$

(Solve for c_n, d_n)

Exterior Dirichlet Problem

Problem 19-2

To find the function $u(r, \theta)$ that satisfies

$$\begin{aligned} \text{PDE:} \quad & u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad && 1 < r < \infty \\ \text{BC:} \quad & u(1, \theta) = g(\theta), && 0 \leqslant \theta < 2\pi. \end{aligned}$$

Problem 19-2 is solved exactly like the interior Dirichlet problem. The only exception is that now we throw out the solutions that are unbounded as r goes to infinity

$$r^n \cos(n\theta), \quad r^n \sin(n\theta), \quad \ln r$$

Exterior Dirichlet Problem (cont.)

Hence, we are left with the solution

$$\boxed{u(r,\theta) = \sum_{n=0}^{\infty} r^{-n} \left[a_n \cos(n\theta) + b_n \sin(n\theta)\right],}$$

where a_n and b_n are exactly as before

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} g(s)ds,$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} g(s)\cos(ns)ds, \qquad b_n = \frac{1}{\pi}\int_0^{2\pi} g(s)\sin(ns)ds$$

Remarks

- The exterior Dirichlet problem for arbitrary radius R

$$\text{PDE:} \quad u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad R < r < \infty$$

$$\text{BC:} \quad u(R, \theta) = g(\theta), \qquad 0 \leqslant \theta < 2\pi.$$

  has the solution

$$u(r, \theta) = \sum_{n=0}^{\infty} \left(\frac{r}{R}\right)^{-n} [a_n \cos(n\theta) + b_n \sin(n\theta)]$$

- The only solution of the 2-D Laplace equation that depend only on r are constants and $\ln r$. The potential $\ln r$ is very important and is called the logarithmic potential.

Laplace's Equation in Spherical Coordinates (Spherical Harmonics)

An important problem is to find the potential inside or outside a sphere when the potential is given on the boundary. Consider, first, the interior problem:

Problem 19-3

To find the function $u(r,\theta,\phi)$ that satisfies

$$\text{PDE:} \quad \left(r^2 u_r\right)_r + \frac{1}{\sin\phi}\left[\sin\phi u_\phi\right]_\phi + \frac{1}{\sin^2\phi} u_{\theta\theta} = 0, \quad 0 < r < 1$$

$$\text{BC:} \quad u(1,\theta,\phi) = g(\theta,\phi), \qquad -\pi \leqslant \theta < \pi, \qquad 0 \leqslant \phi < \pi$$

Remarks

- A typical application of the problem 19-3 would be to find the temperature inside a sphere when the temperature is specified on the boundary.
- Quite often $g(\theta, \phi)$ has a specific form, so that it isn't necessary to solve the problem in its most general form.
- We consider two important cases. One is the case when $g(\theta, \phi)$ is constant, and the other is when it depends only on the angle ϕ (the angle from the north pole).

Special Case 1. ($g(\theta, \phi) = constant$)

- In this case, it is clear that the solution is independent of θ and ϕ, and so Laplace's equation reduces to the ODE

$$(r^2 u_r)_r = 0. \tag{19.3}$$

- The general solution of (19.3) is

$$u(r) = \frac{a}{r} + b$$

- In other words, constants and $\frac{c}{r}$ are the only potentials that depend only on the radial distance from the origin. The potential $\frac{1}{r}$ is called the Newtonian potential.

Special Case 2. ($g(\theta,\phi)$ depends only on ϕ)

In this case, the Dirichlet problem takes the form

Problem 19-3a

To find the function $u(r,\theta,\phi)$ that satisfies

$$\text{PDE:}\quad \left(r^2 u_r\right)_r + \frac{1}{\sin\phi}\left[\sin\phi\, u_\phi\right]_\phi = 0, \quad 0 < r < 1$$

$$\text{BC:}\quad u(1,\theta,\phi) = g(\phi), \qquad 0 \leqslant \phi < \pi$$

Step 1. (Separation of variables)

- We look for solutions of the form

$$u(r, \phi) = R(r)\Phi(\phi)$$

and arrive at the two ODEs

$$r^2 R'' + 2rR' - n(n+1)R = 0 \qquad \text{(Euler's equation)}$$

$$\left[\sin\phi\Phi'\right]' + n(n+1)\sin\phi\Phi = 0 \quad \text{(Legendre's equation)}$$

- The separation constant is chosen to be $n(n+1)$ for convenience; later we will see why this choice is made.

Step 2. (Solving the Euler equation)

- We solve Euler's equation by substituting $R(r) = r^{\alpha}$ in the equation and solving for α. Doing this, we get two values

$$\alpha = \begin{cases} n \\ -(n+1) \end{cases}$$

- Hence, Euler's equation has the general solution

$$R(r) = ar^{n} + br^{-(n+1)}$$

Step 3. (Solving the Legendre equation)

- Making the substitution $x = \cos\phi$ we get the new Legendre equation

  $$(1 - x^2)\frac{d^2\Phi}{dx^2} - 2x\frac{d\Phi}{dx} + n(n+1)\Phi = 0, \quad -1 \leqslant x \leqslant 1.$$

  The idea here is to solve for $\Phi(x)$ and then substitute $x = \cos\phi$ in the solution.

- Legendre's equation is a linear second-order ODE with variable coefficients. One of the difficulties in this equation is that the coefficient $(1 - x^2)$ is zero at the ends of the interval $[-1, 1]$. Equations like this are called singular differential equations and are often solved by the method of Frobenius.

Step 3. (cont.)

- The only bounded solutions of Legendre's equation occur when $n = 0, 1, 2, \dots$ and these solutions are polynomials $P_n(x)$, $-1 \leqslant x \leqslant 1$ (Legendre polynomials)

$$n = 0 \quad P_0(x) = 1$$

$$n = 1 \quad P_1(x) = x$$

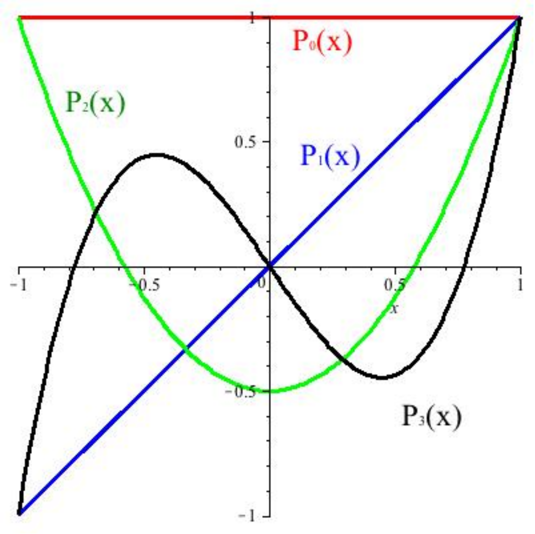
$$n = 2 \quad P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$n = 3 \quad P_3(x) = \frac{1}{2}(5x^2 - 3x)$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$n \quad P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left[(x^2 - 1)^n\right] \quad \text{(Rodrigues' formula)}$$

Legendre Polynomials $P_n(x)$



Step 4. (Combination)

- We now have that the bounded solutions of

$$r^2 R'' + 2rR' - n(n+1)R = 0 \qquad 0 < r < 1$$
$$\left[\sin\phi \Phi'\right]' + n(n+1)\sin\phi \Phi = 0 \qquad -\pi \leqslant \phi \leqslant \pi$$

are

$$R(r) = ar^n$$
$$\Phi(\phi) = aP_n(\cos\phi)$$

- Therefore,

$$u(r, \phi) = \sum_{n=0}^{\infty} a_n r^n P_n(\cos\phi). \tag{19.4}$$

Step 5. (Substituting into BC)

- Substituting solution (19.4) into the BC gives

$$\sum_{n=0}^{\infty} a_n P_n(\cos\phi) = g(\phi) \tag{19.5}$$

- Observe that the Legendre polynomials are orthogonal on $[-1, 1]$.

Step 5. (cont.)

So, if we multiply each side of (19.5) by $P_m(\cos\phi)\sin\phi$ and integrate ϕ from 0 to π, we get

$$\int_0^\pi g(\phi)P_m(\cos\phi)\sin\phi d\phi = \sum_{n=0}^\infty a_n \int_0^\pi P_n(\cos\phi)P_m(\cos\phi)\sin\phi d\phi$$

$$= \sum_{n=0}^\infty a_n \int_{-1}^1 P_n(x)P_m(x)dx$$

$$= \begin{cases} 0, & n \neq m \\ \dfrac{2a_m}{2m+1}, & m = n \end{cases}$$

Step 5. (cont.)

- Hence

$$a_n = \frac{2n+1}{2} \int_0^\pi g(\phi) P_n(\cos\phi) \sin\phi \, d\phi$$

and the solution to our Dirichlet problem 19-3a is

$$u(r, \phi) = \sum_{n=0}^{\infty} a_n r^n P_n(\cos\phi)$$

Remarks

- The solution of the exterior Dirichlet problem

$$\begin{aligned} \text{PDE:} \quad & \Delta u = 0, & 1 < r < \infty \\ \text{BC:} \quad & u(1, \theta, \phi) = g(\phi), & 0 \leqslant \phi < \pi \end{aligned}$$

is

$$u(r, \phi) = \sum_{n=0}^{\infty} \frac{b_n}{r^{n+1}} P_n(\cos\phi),$$

where

$$b_n = \frac{2n+1}{2} \int\limits_0^{\pi} g(\phi) P_n(\cos\phi) \sin\phi d\phi.$$

Remarks (cont.)

- For example, the BC $g(\phi) = 3$ would yield for the solution of the exterior problem
$$u(r, \phi) = \frac{3}{r}.$$
Note that in this problem (in 3D!!!), the solution goes to zero, while in two dimensions, the exterior solution with constant BC was itself a constant.