

PDE and Boundary-Value Problems

Winter Term 2015/2016

Lecture 2

Universität des Saarlandes

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Purpose of Lesson

- To continue the discussion about classification of 2nd order PDEs and define the normal (canonical) forms of 2nd order PDEs in two variables.
- To introduce the notions of BVPs and classical solutions.

Normal forms of 2nd order PDEs in two independent variables:

Using a suitable transformation of independent variables

$$\xi = \xi(x, y), \quad \eta = \eta(x, y)$$

we can always reduce equation

$$a(x, y)u_{xx} + b(x, y)u_{xy} + c(x, y)u_{yy} + f(x, y, u, u_x, u_y) = 0$$

to one of the following three NORMAL FORMS:

- for **hyperbolic** equations

$$u_{\xi\eta} = F(\xi, \eta, u, u_{\xi}, u_{\eta}), \quad \text{or} \quad u_{\xi\xi} - u_{\eta\eta} = F(\xi, \eta, u, u_{\xi}, u_{\eta});$$

- for **parabolic** equations

$$u_{\eta\eta} = F(\xi, \eta, u, u_{\xi}, u_{\eta}),$$

where F **must** depend on u_{ξ} : otherwise the equation degenerates into an ODE;

- for **elliptic** equations

$$u_{\xi\xi} + u_{\eta\eta} = F(\xi, \eta, u, u_{\xi}, u_{\eta}).$$

The classification (elliptic, parabolic etc.) can be extended to equations depending on more than 2 variables.

Consider the 2nd order PDE depending on n variables,

$$\sum_{i=1}^n \sum_{j=1}^n a_{ij} u_{x_i x_j} + \sum_{i=1}^n b_i u_{x_i} + cu + g = 0.$$

The coefficient matrix (a_{ij}) should be symmetrized because

$$\frac{\partial^2}{\partial x_i \partial x_j} = \frac{\partial^2}{\partial x_j \partial x_i}, \quad \text{for any } i \text{ and } j \text{ in } [1, n].$$

The classification is as follows:

- **hyperbolic** for $(Z = 0 \text{ and } P = 1)$ or $(Z = 0 \text{ and } P = n - 1)$
- **parabolic** for $Z > 0$ ($\Leftrightarrow \det(a_{ij}) = 0$)
- **elliptic** for $(Z = 0 \text{ and } P = n)$ or $(Z = 0 \text{ and } P = 0)$
- **ultra-hyperbolic** for $(Z = 0 \text{ and } 1 < P < n - 1)$

where

$Z =$ number of **zero** eigenvalues (a_{ij}) ,

$P =$ number of **strictly positive** eigenvalues of (a_{ij}) .

What are Boundary Value and Initial Value Problems?

PDEs can have many very different solutions. For example, the Laplace equation

$$u_{xx} + u_{yy} = 0$$

is solved by

$$u = x^2 - y^2, \quad u = e^x \cos y, \quad u = \ln(x^2 + y^2).$$

General solutions of higher-order PDEs are often **difficult to find** and **hard to use**.

A **unique** solution modelling a given processes or phenomenon can be specified by additional **constraints** imposed on the boundary of the region in space (**boundary condition**) or at some time (**initial conditions**).

Remarks

- If the number of such constraints is **too large**, the problem will be **overdetermined** and will not have any solutions.
- If the number of constraints is **too small**, the problem will have more than one solution.

If a differential equation has been given together with all necessary boundary and/or initial conditions, it is said that a **boundary value** or **initial value problem** has been formulated.

Common IVPs, formulated in a region $\mathbb{R}^n \times [0; +\infty)$ are as follows:

- **The Cauchy problem:** Determine a function u such that

$$\left\{ \begin{array}{l} \text{PDE in } \mathbb{R}^n \times [0, +\infty) \\ u|_{t=0} = \varphi(\mathbf{x}) \end{array} \right. \quad \text{or} \quad \left\{ \begin{array}{l} \text{PDE in } \mathbb{R}^n \times [0, +\infty) \\ u|_{t=0} = \varphi(\mathbf{x}) \\ u_t|_{t=0} = \psi(\mathbf{x}), \end{array} \right.$$

where φ and ψ are given functions, defined in \mathbb{R}^n .

Common BVPs, formulated in a region Ω in space, (Ω has the boundary $\partial\Omega$) are as follows:

- **The Dirichlet problem:** Determine a function u such that

$$\begin{cases} \text{PDE} & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

where φ is a given function defined on $\partial\Omega$.

- **The Neumann problem:** Determine a function u such that

$$\begin{cases} \text{PDE} & \text{in } \Omega \\ \frac{\partial u}{\partial n} = \psi & \text{on } \partial\Omega, \end{cases}$$

where ψ is a given function defined on $\partial\Omega$ and $\frac{\partial u}{\partial n}$ is the normal derivative.

We say that a given BVP is **well-posed** if

- the BVP in fact has a solution;
- this solution is unique;
- the solution depends continuously on the data given in the problem.

Remark

The last condition is particularly important for problems arising from physical applications: we would prefer that our (unique) solution changes only a little when the conditions specifying the problem change a little.

How regular should be a solution?

Should we ask, for example, that a solution u of our BVP must be real analytic or at least infinitely differentiable?

This might be desirable, but perhaps we are asking **too much**.

It would be more practical to require a solution of the BVP with PDE of **order k** to be at least **k times continuously differentiable**.

Then at least all the derivatives which appear in the statement of the PDE will exist and be continuous, although maybe certain higher derivatives will not exist.

We call a solution with this much smoothness a **classical solution**.