PDE and Boundary-Value Problems Winter Term 2015/2016

Lecture 20

Saarland University

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Purpose of Lesson

- To derive the fundamental solution of the Laplace equation and discuss how to solve with its help the Poisson equation (nonhomogeneous Laplace equation).
- To show how a nonhomogeneous Dirichlet problem can be solved by the Green's function approach (the impulse-responce function).
- To derive Green's functions for a half-space and for a ball.
- To show how a PDE can be changed to a system of algebraic equations by replacing the partial derivatives in the differential equation with their finite-difference approximations. The system of algebraic equations can then be solved numerically by an iterative process in order to obtain an approximate solution to the PDE.

Fundamental Solution

Problem 20-1

To find a function $u(x)$ that satisfies

$$\Delta u = 0 \qquad x \in \mathbb{R}^n$$

- We attempt to find a solution u of Laplace's equation in $\mathbb{R}^n$, having the form

$$u(x) = v(r),$$

where $r = |x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$.

- It is clear that $\Delta u = 0$ if and only if that

$$v''(r) + \frac{n-1}{r} v'(r) = 0.$$

- If $r > 0$ we have

$$v(r) = \begin{cases} a \ln r + b & (n = 2) \\ \dfrac{a}{r^{n-2}} + b & (n \geqslant 3), \end{cases}$$

where a and b are constants.

The above consideration motivate the following

Definition

The function

$$\Phi(x) := \begin{cases} -\dfrac{1}{2\pi} \ln |x|, & (n = 2) \\ \dfrac{1}{n(n-2)\omega(n)} \dfrac{1}{|x|^{n-2}} & (n \geqslant 3), \end{cases}$$

defined for $x \in \mathbb{R}^n$, $x \neq 0$, is the fundamental solution of Laplace's equation.

- $\omega(n)$ denotes the volume of the unit ball in $\mathbb{R}^n$.
- $|D\Phi(x)| \leqslant \dfrac{C}{|x|^{n-1}}, \quad |D^2\Phi(x)| \leqslant \dfrac{C}{|x|^n}, \quad (x \neq 0)$

Theorem (Solving Poisson's equation)

Let $f : \mathbb{R}^n \to \mathbb{R}$ be twice contionuous differentiable with compact support, and let u satisfy

$$u(x) = \int\limits_{\mathbb{R}^n} \Phi(x-y)f(y)dy$$

$$= \begin{cases} -\dfrac{1}{2\pi} \displaystyle\int\limits_{\mathbb{R}^2} \ln|x-y| f(y)dy & (n=2) \\ \dfrac{1}{n(n-2)\omega(n)} \displaystyle\int\limits_{\mathbb{R}^n} \dfrac{f(y)}{|x-y|^{n-2}}dy & (n \geqslant 3) \end{cases}$$

Then $u \in C^2(\mathbb{R}^n)$ and $-\Delta u = f \quad \text{in } \mathbb{R}^n$.

Remarks

- We cannot just compute

$$\Delta u(x) = \int\limits_{\mathbb{R}^n} \Delta_x \Phi(x-y) f(y) dy = 0.$$

- Indeed, $D^2\Phi(x-y)$ is not summable near the singularity at $y = x$, and so the differentiation under the integral sign is unjustified (and incorrect).
- We must proceed more carefully in calculating Δu.

A Nonhomogeneous Dirichlet Problem (Green's Function)

Problem 20-2

To find a function $u(x)$ that satisfies

$$\text{PDE:} \quad -\Delta u = f, \quad x \in U \subset \mathbb{R}^n, \quad U - \text{open, bounded, } \partial U \in C^1$$

$$\text{BC:} \quad u = g, \quad x \in \partial U$$

We propose to obtain a general representation formula for the solution of Problem 20-2.

Derivation of Green's function

- Fix $x \in U$, choose $\varepsilon > 0$ so small that $B_\varepsilon(x) \subset U$.
- Applying Green's formula

$$\int_\Omega [u\Delta v - v\Delta u]\, dx = \int_{\partial\Omega} \left[u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right] dS$$

on the region $U_\varepsilon := U - B_\varepsilon(x)$ to $u(y)$ and $\Phi(y-x)$ we get

$$\int_{U_\varepsilon} [u(y)\Delta\Phi(y-x) - \Phi(y-x)\Delta u(y)]dy$$

$$= \int_{\partial U_\varepsilon} \left[u(y)\frac{\partial \Phi}{\partial n}(y-x) - \Phi(y-x)\frac{\partial u}{\partial n} \right] dS(y)$$

Derivation of Green's function (cont.)

- We observe that

1. $\Delta\Phi(y-x) = 0$ for $y \neq x$;
2. $$\Big|\int\limits_{\partial B_\varepsilon(x)} \Phi(y-x)\frac{\partial u}{\partial n}(y)dS(y)\Big| \leqslant C\varepsilon^{n-1}\max_{\partial B_\varepsilon(x)}|\Phi| = o(1)$$
   as $\varepsilon \to 0$;
3. $$\int\limits_{\partial B_\varepsilon(x)} u(y)\frac{\partial \Phi}{\partial n}(y-x)dS(y) = \fint\limits_{\partial B_\varepsilon(x)} u(y)dS(y) \to u(x)$$
   as $\varepsilon \to 0$;

Derivation of Green's function (cont.)

- Hence our sending $\varepsilon \to 0$ yields the formula:

$$u(x) = \int\limits_{\partial U} \left[\Phi(y-x) \frac{\partial u}{\partial n}(y) - u(y) \frac{\partial \Phi}{\partial n}(y-x) \right] dS(y) - \int\limits_{U} \Phi(y-x) \Delta u(y) dy$$

The above identity is valid for any point $x \in U$ and any function $u \in C^2(\overline{U})$

Derivation of Green's function (cont.)

$$u(x) = \int_{\partial U} \left[\Phi(y-x) \frac{\partial u}{\partial n}(y) - u(y) \frac{\partial \Phi}{\partial n}(y-x) \right] dS(y) - \int_{U} \Phi(y-x) \Delta u(y) dy \tag{20.1}$$

Remarks

- If we apply formula (20.1) to problem 20-2, we see that the normal derivative $\frac{\partial u}{\partial n}$ along ∂U is unknown to us.
- We must somehow modify (20.1) to remove this term.

Derivation of Green's function (cont.)

The idea is now to introduce for fixed x a corrector function $\phi^x = \phi^x(y)$, solving the BVP:

Problem 20-3

To find a function $\phi^x(y)$ that satisfies

$$\text{PDE:} \quad \Delta\phi^x = 0, \qquad y \in U$$

$$\text{BC:} \quad \phi^x = \Phi(y - x), \qquad y \in \partial U$$

Derivation of Green's function (cont.)

Applying Green's formula once more, we compute

$$-\int\limits_{U} \phi^x(y)\Delta u(y)dy = \int\limits_{\partial U} \left[u(y)\frac{\partial \phi^x}{\partial n}(y) - \phi^x(y)\frac{\partial u}{\partial n}(y)\right] dS(y)$$

$$= \int\limits_{\partial U} \left[u(y)\frac{\partial \phi^x}{\partial n}(y) - \Phi(y-x)\frac{\partial u}{\partial n}(y)\right] dS(y)$$

We introduce next this

Definition

Green's function for the region U is

$$G(x,y) := \Phi(y-x) - \phi^x(y) \qquad (x, y \in U, \quad x \neq y).$$

Adopting this terminilogy, we find from (20.1) and the above identity for ϕ^x the formula

$$u(x) = -\int\limits_{\partial U} u(y)\frac{\partial G}{\partial n}(x,y)dS(y) - \int\limits_{U} G(x,y)\Delta u(y)dy, \quad (x \in U) \quad (20.2)$$

where

$$\frac{\partial G}{\partial n}(x,y) = n(y) \cdot D_y G(x,y)$$

is the outer normal derivative of G with respect to the variable y.

Theorem (Representation formula using Green's function)

If $u \in C^2(\overline{U})$ solves problem 20-2, then

$$u(x) = -\int\limits_{\partial U} g(y)\frac{\partial G}{\partial n}(x,y)dS(y) + \int\limits_{U} f(y)G(x,y)dy \ (x \in U)$$

- To construct Green's function G for the given domain U is in general a difficult matter, and can be done only when U has simple geometry
- We will build Green's functions for two regions with simple geometry, namely the half-space $\mathbb{R}^n_+$ and the unit ball $B_1(0)$.

Theorem (Representation formula using Green's function)

If $u \in C^2(\overline{U})$ solves the problem

$$\text{PDE:} \quad -\Delta u = f, \quad x \in U \subset \mathbb{R}^n, \quad U - \text{open, bounded, } \partial U \in C^1$$

$$\text{BC:} \quad u = g, \quad x \in \partial U$$

then

$$\boxed{u(x) = -\int\limits_{\partial U} g(y) \frac{\partial G}{\partial n}(x,y) dS(y) + \int\limits_{U} f(y) G(x,y) dy \; (x \in U)},$$

where

$$G(x,y) := \Phi(y-x) - \phi^x(y) \qquad (x, y \in U, \quad x \neq y).$$

Green's Function for a Half-Space

- We set for $x, y \in \mathbb{R}^n_+$

$$\phi^x(y) := \Phi(y - \tilde{x}) = \Phi(y_1 - x_1, \dots, y_{n-1} - x_{n-1}, y_n + x_n).$$

The idea is that the corrector ϕ^x is built from Φ by „reflecting the singularity" from $x \in \mathbb{R}^n_+$ to $\tilde{x} \notin \mathbb{R}^n_+$.

- If $y \in \partial\mathbb{R}^n_+$ then

$$\phi^x(y) = \Phi(y - x),$$

and thus

$$\begin{cases} \Delta\phi^x = 0 & \text{in } \mathbb{R}^n_+ \\ \phi^x = \Phi(y - x) & \text{on } \partial\mathbb{R}^n_+ \end{cases}$$

as required.

Green's Function for a Half-Space (cont.)

- Therefore, Green's function for the half-space $\mathbb{R}^n_+$ is

$$G(x,y) := \Phi(y-x) - \Phi(y-\tilde{x}), \qquad (x, y \in \mathbb{R}^n_+,\ x \neq y)$$

- It is evident that

$$\begin{aligned} \frac{\partial G}{\partial y_n}(x,y) &= \frac{\partial \Phi}{\partial y_n}(y-x) - \frac{\partial \Phi}{\partial y_n}(y-\tilde{x}) \\ &= \frac{-1}{n\omega(n)}\left[\frac{y_n - x_n}{|y-x|^n} - \frac{y_n + x_n}{|y-\tilde{x}|^n}\right] \end{aligned}$$

- So, if $y \in \partial\mathbb{R}^n_+$

$$\frac{\partial G}{\partial n}(x,y) = -\frac{\partial G}{\partial y_n}(x,y) = -\frac{-2x_n}{n\omega(n)}\frac{1}{|x-y|^n}.$$

Green's function for a half-space (cont.)

Suppose now u solves the BVP

Problem 20-4

To find a function $u(x)$ that satisfies

$$\text{PDE:} \quad \Delta u = 0, \qquad x \in \mathbb{R}^n_+$$

$$\text{BC:} \quad u = g, \qquad x \in \partial\mathbb{R}^n_+$$

We expect

$$u(x) = \frac{2x_n}{n\omega(n)} \int\limits_{\partial\mathbb{R}^n_+} \frac{g(y)}{|x-y|^n dy} \quad (x \in \mathbb{R}^n_+) \tag{20.3}$$

to be a representation formula for our solution.

Green's Function for a Half-Space (cont.)

Remark

We must check directly that formula (20.3) provides us with a solution of the BVP 20-4, i.e., we must check that

$$u \in C^2(\overline{\mathbb{R}^n_+}),$$

and

$$\Delta u = 0 \quad \text{in } \mathbb{R}^n_+,$$

and

$$\lim_{\mathbb{R}^n_+ \ni x \to x^0} u(x) = g(x^0) \quad \text{for each point } x^0 \in \partial\mathbb{R}^n_+.$$

Green's Function for a Ball

We solve problem for the unit ball, i.e.,

$$U = B_1(0) = \{x \in \mathbb{R}^n : \; |x| < 1\}$$

- We set for $x, y \in B_1(0)$

$$\phi^x(y) := \Phi\left(|x|(y - \tilde{x})\right), \qquad \tilde{x} = \frac{x}{|x|^2}$$

Again, the idea is to „invert the singularity“ from $x \in B_1(0)$ to $\tilde{x} \notin B_1(0)$.

Green's Function for a Ball (cont.)

- Assume for the moment $n \geqslant 3$.
  1. The mapping $y \mapsto \Phi(y - \tilde{x})$ is harmonic for $y \neq \tilde{x}$
  2. Thus $y \mapsto |x|^{2-n}\Phi(y - \tilde{x})$ is also harmonic for $y \neq \tilde{x}$

  Therefore, $\phi^x(y) := \Phi(|x|(y - \tilde{x})$ is harmonic in $B_1(0)$.
- If $y \in \partial B_1(0)$ and $x \neq 0$, then

$$
\begin{aligned}
|x|^2|y - \tilde{x}|^2 &= |x|^2\left(|y|^2 - \frac{2y \cdot x}{|x|^2} + \frac{1}{|x|^2}\right) \\
&= |x|^2 - 2y \cdot x + 1 = |x - y|^2.
\end{aligned}
$$

  Thus, $(|x||y - \tilde{x}|)^{-(n-2)} = |x - y|^{-(n-2)}$. Consequently

$$
\phi^x(y) = \Phi(y - x) \quad (y \in \partial B_1(0))
$$

  as required.

Green's Function for a Ball (cont.)

- Therefore, Green's function for the unit ball $B_1(0)$ is

$$G(x,y) := \Phi(y-x) - \Phi(|x|(y-\tilde{x}) \quad (x, y \in B_1(0), x \neq y)$$

- If $y \in \partial B_1(0)$ then

$$\begin{aligned}
\frac{\partial G}{\partial n}(x,y) &= \sum_{i=1}^{n} y_i \frac{\partial G}{\partial y_i}(x,y) \\
&= \frac{-1}{n\omega(n)} \frac{1}{|x-y|^n} \sum_{i=1}^{n} y_i \left((y_i - x_i) - y_i |x|^2 + x_i \right) \\
&= \frac{-1}{n\omega(n)} \frac{1 - |x|^2}{|x-y|^n}.
\end{aligned}$$

Green's Function for a Ball (cont.)

Suppose now u solves the BVP

Problem 20-5

To find a function $u(x)$ that satisfies

$$\text{PDE:} \quad \Delta u = 0, \qquad x \in B_1(0)$$

$$\text{BC:} \quad u = g, \qquad x \in \partial B_1(0)$$

We expect

$$u(x) = \frac{1-|x|^2}{n\omega(n)} \int\limits_{\partial B_1(0)} \frac{g(y)}{|x-y|^n dS(y)} \quad (x \in B_1(0)) \tag{20.4}$$

to be a representation formula for our solution.

Chapter 5. Numerical and Approximate Methods

- So far, we have studied several techniques for solving linear PDEs. However, most of the equations we've attackes were reasonably simple, had reasonably simple BCs, and had reasonably shaped domains.
- But many problems cannot be simplified to fit this general mold and must be solved by numerical approximations.
- To begin, we introduce the idea of finite differences. We then show how to use these finite differences to solve a Dirichlet problem inside a square.

Finite-Difference Approximations

- First, we recall the Taylor series expansion of a function $f(x)$

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2!}h^2 + \dots$$

- If we truncate this series after two terms, we have the approximation

$$f(x+h) \cong f(x) + f'(x)h$$

Hence, we can solve for $f'(x)$

$$f'(x) \cong \frac{f(x+h) - f(x)}{h}$$

which is called the forward-difference approximation to the first derivative $f'(x)$.

Finite-Difference Approximations (cont.)

- We could also replace h by $-h$ in the Taylor series and arrive at the backward-difference approximation

$$f'(x) \cong \frac{f(x) - f(x-h)}{h}$$

or by subtracting

$$f(x-h) \cong f(x) - f'(x)h$$

from

$$f(x+h) \cong f(x) + f'(x)h$$

we can obtain the central-difference approximation

$$f'(x) \cong \frac{1}{2h}\left[f(x+h) - f(x-h)\right].$$

Finite-Difference Approximations (cont.)

- By retaining another term in the Taylor series, this type of analysis can be extended to arrive at the central-difference approximation of the second derivative $f''(x)$

$$f''(x) \cong \frac{1}{h^2}\left[f(x+h) - 2f(x) + f(x-h)\right].$$

- We now extend the finite-difference approximations to partial derivatives. If we begin with the Taylor series expansion in two variables

$$u(x+h,y) = u(x,y) + u_x(x,y)h + u_{xx}(x,y)\frac{h^2}{2!} + \dots$$
$$u(x-h,y) = u(x,y) - u_x(x,y)h + u_{xx}(x,y)\frac{h^2}{2!} - \dots$$

we can deduce the following:

$$u_x(x,y) \cong \frac{u(x+h,y)-u(x,y)}{h} \quad \text{(Forward difference)}$$
$$u_{xx}(x,y) \cong \frac{1}{h^2}\left[u(x+h,y) - 2u(x,y) + u(x-h,y)\right]$$
$$u_y(x,y) \cong \frac{1}{k}\left[u(x,y+k) - u(x,y)\right]$$
$$u_{yy}(x,y) \cong \frac{1}{k^2}\left[u(x,y+k) - 2u(x,y) + u(x,y-k)\right].$$

Remarks

- Which approximation of partial derivatives is used (forward, central, or backward) depends on the problem.
- We will consider the central-difference approximation.

Dirichlet Problem Solved by the Finite-Difference Method

To illustrate how to use these finite-difference approximations, we consider the simple Dirichlet problem.

Problem 20-6

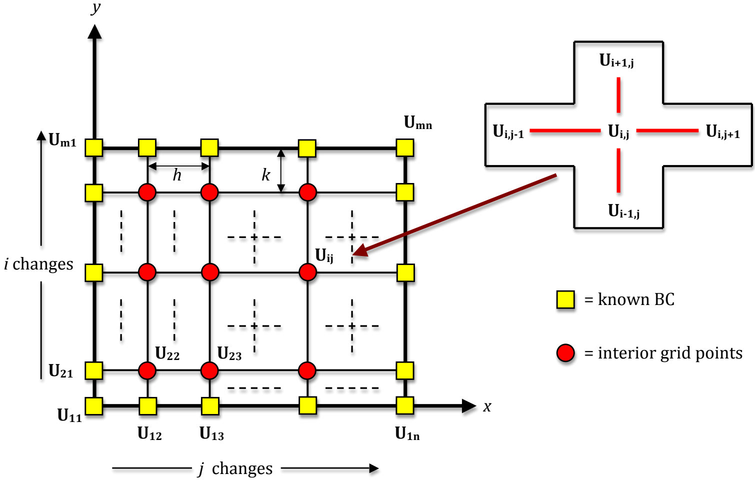
To find a function $u(x, y)$ that satisfies

PDE: $$u_{xx} + u_{yy} = 0, \qquad 0 < x < 1, \qquad 0 < y < 1$$

BCs: $$\begin{cases} u(x, y) = 0 & \text{On the top and sides of the square} \\ u(x, 0) = \sin(\pi x) & 0 \leqslant x \leqslant 1 \end{cases}$$

We begin with the drawing the grid system on the *xy*-plane.

Grid lines for the Dirichlet problem inside a square



It is convenient to use the following notation:

$$
\begin{aligned}
u(x,y) &= u_{i,j}\\
u(x,y+k) &= u_{i+1,j}\\
u(x,y-k) &= u_{i-1,j}\\
u(x+h,y) &= u_{i,j+1}\\
u(x-h,y) &= u_{i,j-1}\\
u_x(x,y) &= \frac{1}{2h}(u_{i,j+1}-u_{i,j-1})\\
u_y(x,y) &= \frac{1}{2k}(u_{i+1,j}-u_{i-1,j})\\
u_{xx}(x,y) &= \frac{1}{h^2}(u_{i,j+1}-2u_{i,j}+u_{i,j-1})\\
u_{yy}(x,y) &= \frac{1}{k^2}(u_{i+1,j}-2u_{i,j}+u_{i-1,j})
\end{aligned}
$$

- Our strategy for solving the Dirichlet problem 20-6 is to replace the partial derivatives in Laplace's equation

$$u_{xx} + u_{yy} = 0$$

by their finite-difference approximations.

- Using the compact notation $u_{i,j}$, we have the following difference equation:

$$\Delta u = \frac{1}{h^2}(u_{i,j+1} - 2u_{i,j} + u_{i,j-1}) + \frac{1}{k^2}(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) = 0.$$

- By letting the two discretization sizes h and k be the same, Laplace's equation is replaced by

$$(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}) = 0$$

or solving for $u_{i,j}$

$$\boxed{u_{i,j} = \frac{1}{4}\left(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}\right)} \qquad (20.5)$$

Remarks

1. $u_{i,j}$ stands for the solution at the interior grid points.
2. Equation (20.5) says that we can approximate the solution $u_{i,j}$ by averaging the solution at four neighboring grid points.

Numerical Algorithm for Solving the Dirichlet Problem (Liebmann's Method)

1. Seek the solution $u_{i,j}$ at the interior grid points by setting them equal to the average of all the BCs (reasonable start).
2. Systematically run over al the interior grid points, replacing the old estimates by the average of its four neighbors.

Remarks

1. It doesn't make much difference in what order this process is carried out, but, generally, it si done in a row by row (or colunm by colunm) manner.
2. After a few iterations, this process will converge to an approximate solution of the problem.
3. The rate of convergence is generally slow but can be speeded up in a number of ways.

Remarks

- If we made our discretization sizes h and k smaller (so that we had more grid points), the analysis would be similar except that the system of obtained algebraic equations would be larger.
- In general, the number of equations will be equal to the number of interior grid points.
- To solve the Neumann problem where there are derivatives on the boundary we must also replace these derivatives by some finite difference approximation.
- We can also solve equations with variable coefiicients and nonhomogeneous equations by the finite-difference method.

Remarks (cont.)

- If the domain of the problem is an irregularly shaped region, we can overlay the region with grid lines and then approximate the solution at nearby grid points by interpolation the BCs.