PDE and Boundary-Value Problems
Winter Term 2015/2016

Lecture 3

Universität des Saarlandes

30. Oktober 2015

Purpose of Lesson

- To show how parabolic PDEs are used to model heat-flow and diffusion-type problems. To discuss the physical meaning of different terms (such as u_t, u_x, u_{xx}, and u) and present a few examples of parabolic equations.
- To give an intuitive feeling for parabolic-type problems.
- To discuss first of three important types of BCs:
  1. temperature specified on the boundary.

Chapter 2. Parabolic (Diffusion)-Type Problems.

We introduce a simple physical problem and show how it can be described by means of a mathematical model (model involves PDEs).

Then we complicate the problem and show how new PDEs can describe the new situations.

Remark

Today we will not derive or solve PDEs. It will be done later.

A simple heat-flow experiment

Suppose we have the following simple experiment that we break into steps:

1. We start with a reasonable long (say $L = 2$m) rod 2 cm in diameter. The heat can flow in and out of the rod at the end, but not across the lateral boundary.

2. Next, we place this rod in an enviroment whose temperature is fixed at $T_0 = 10^\circ C$ for a sufficiently long time. The temperature of the entire rod comes to a steady state temperature similar to the enviroment.

3. We take the rod out of the enviroment at the time that we call $t = 0$ and attach two *temperature elements* to the end of the rod. These elements keep the ends at specific temperature $T_1 = 0^\circ C$ and $T_2 = 50^\circ C$.

4. Now we monitor the temperature profile of the rod on some type of display.

The mathematical model of the heat-flow experiment

The description of our heat-transfer problem requires three types of equations

1. The PDE describing the physical phenomenon of heat flow.
2. The boundary conditions describing the physical nature of our problem on the boundaries
3. The initial conditions describing the physical phenomenon at the start of the experiment.

The heat equation

The basic equation of *one-dimensional* heat flow is the relationship

$$u_t = \alpha^2 u_{xx}, \qquad 0 < x < L, \quad 0 < t < \infty \tag{3.1}$$

which relates the quantities

u_t = the rate of change in temperature w.r.t. time

and

u_{xx} = the concavity of the temperature profile $u(x, t)$.

Remarks

- u_{xx} compares the temperature at one point to the temperature at neighboring points.
- Equation (3.1) will be derived later from the basic conservation of heat equation.
- Eq. (3.1) says that the temperature $u(x, t)$ at some point x and at some moment t is increasing ($u_t > 0$) or decreasing ($u_t < 0$) according to whether u_{xx} is positive or negative.
- The proportionality constant α^2 is a property of the material.

How u_{xx} can be interpreted to measure heat flow?

We approximate u_{xx} by the difference quotient

$$u_{xx}(x,t) \cong \frac{1}{(\Delta x)^2}\left[u(x+\Delta x,t) - 2u(x,t) + u(x-\Delta x,t)\right]$$
$$= \frac{2}{(\Delta x)^2}\left[\frac{u(x+\Delta x,t)+u(x-\Delta x,t)}{2} - u(x,t)\right].$$

There is the following interpretation of u_{xx}:

- If $u(x,t) <$ average of the two neighboring temperatures, then $u_{xx} > 0$ (the net flow of the heat into x is positive).
- If $u(x,t) =$ average of the two neighboring temperatures, then $u_{xx} = 0$ (the net flow of the heat into x is zero).
- If $u(x,t) >$ average of the two neighboring temperatures, then $u_{xx} < 0$ (the net flow of the heat into x is negative).

Boundary conditions

All physical problems have boundaries of some kind, so we must describe mathematically what goes on there in order to adequately describe the problem.

Boundary conditions in our experiment:

$$\begin{cases} u(0,t) = T_1 \\ u(L,t) = T_2 \end{cases} \qquad 0 < t < \infty. \tag{3.2}$$

Remark

Temperature u was fixed for all time at T_1 and T_2 at the two ends $x = 0$ and $x = L$.

Initial conditions

All physical problems must start from some value of time (generally called $t = 0$), so we must specify the physical apparatus at this time.

Initial conditions in our experiment

$$u(x, 0) = T_0 \qquad 0 \leqslant x \leqslant L. \tag{3.3}$$

Remark

We started monitoring the rod temperature from the time the rod had achieved a constant temperature of T_0.

We have described the experiment.

Writing (3.1)-(3.3) together, we get an initial boundary value problem:

$$\begin{cases} u_t = \alpha^2 u_{xx} & 0 < x < L, \quad 0 < t < +\infty \\ u(0,t) = T_1 & 0 < t < +\infty \\ u(L,t) = T_2 & 0 < t < +\infty \\ u(x,0) = T_0 & 0 \leqslant x \leqslant L. \end{cases}$$

Lateral heat loss proportional to the temperature difference

The equation

$$u_t = \alpha^2 u_{xx} - \beta (u - u_0) \qquad \beta > 0$$

decribes heat flow in the rod with both

1. diffusion $\alpha^2 u_{xx}$ along the road and
2. heat loss (or gain) across the lateral sides of the rod.

Remark

Heat loss ($u > u_0$) or gain ($u < u_0$) is proportional to the difference between the temperature $u(x, t)$ of the rod and the surrounding medium u_0.

Internal heat source

The nonhomogeneous equation

$$u_t = \alpha^2 u_{xx} + f(x, t)$$

corresponds to the case where the rod is being supplied with an internal heat source (everywhere along the rod and for all time).

Diffusion-convection equation

Suppose a pollutant is carried along in a stream moving with velocity ν.

Let $u(x,t)$ be a concentration of the substance. The rate of change u_t is measured by the diffusion-convection equation

$$u_t = \alpha^2 u_{xx} - \nu u_x,$$

where

$$\alpha^2 u_{xx} \text{ is the diffusion contribution}$$

and

$$-\nu u_x \text{ is the convection component.}$$

Heat equation with variable coefficient

The heat equation

$$u_t = \alpha^2(x) u_{xx}$$

corresponds to a problem where the diffusion within the rod depends on x (the material is nonhomogeneous).

Boundary conditions for parabolic-type problems

Three basic types of boundary conditions can occur for heat-flow problems:

1. $u = g(t);$
2. $\dfrac{\partial u}{\partial n} + \lambda u = g(t).$
3. $\dfrac{\partial u}{\partial n} = g(t).$

Here n ist the outward normal direction to the boundary.

Type 1 BC (Temperature specified on the boundary)

$$\begin{cases} u(0,t) = g_1(t) \\ u(L,t) = g_2(t) \end{cases}$$

(The same heat-flow experiment as in the beginning of the lecture)

Remarks

- Problems with BCs of this kind are fairly common.
- It may be that the goal of the problem is to find the boundary temperatures (boundary control) $g_1(t)$ and $g_2(t)$ that will force the temperature to behave in a suitable manner.
- Similar types of BCs also apply to higher dimensional domains.

Type 1 BC (Temperature specified on the boundary)

Example

In two dimensions we could imagine the problem of finding the temperature inside the circular disc (of radius R) when the boundary temperature is specified in polar coordinates to be

$$u(R, \theta, t) = \cos t \sin \theta.$$

Type 1 BC (Temperature specified on the boundary)

$$u(R, \theta, t) = \cos t \sin \theta.$$

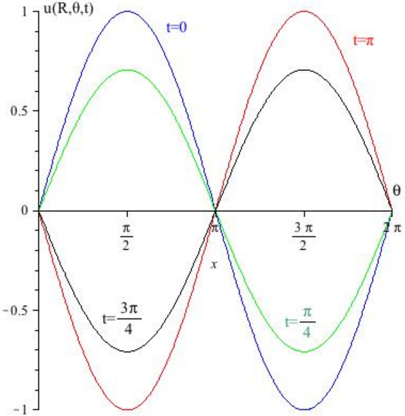

Figure 1: Oscillating boundary temperature