

# PDE and Boundary-Value Problems

## Winter Term 2015/2016

### Lecture 6

Saarland University

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## Purpose of Lesson

- To show how problems with **nonhomogeneous** BCs can be solved by transforming them into others with zero BCs.
- To show how more complicated heat-flow problems can be solved by separation of variables.

# Transforming Nonhomogeneous BCs into Homogeneous Ones

## Problem 6-1

Consider heat flow in an insulated rod where two ends are kept at constant temperature  $k_1$  and  $k_2$ ; that is,

$$\text{PDE: } u_t = \alpha^2 u_{xx}, \quad 0 < x < L, \quad 0 < t < \infty$$

$$\text{BCs: } \begin{cases} u(0, t) = k_1 \\ u(L, t) = k_2 \end{cases}, \quad 0 < t < \infty$$

$$\text{IC: } u(x, 0) = \phi(x), \quad 0 \leq x \leq L.$$

The difficulty here is that since the BCs are not homogeneous, we cannot solve this problem by separation of variables.

## Transforming Nonhomogeneous BCs into Homogeneous Ones

- It is obvious that the solution of problem 6-1 will have a steady-state solution (solution when  $t = \infty$ ) that varies **linearly** (in  $x$ ) between the boundary temperatures  $k_1$  and  $k_2$ .
- It seems reasonable to think of our temperature  $u(x, t)$  as the sum of two parts

$$u(x, t) = \text{steady state} + \text{transient},$$

where **steady state** is eventual solution for large times, while **transient** is a part of the solution that depends on the IC (and will go to zero). So,

$$u(x, t) = \left[ k_1 + \frac{x}{L}(k_2 - k_1) \right] + U(x, t).$$

# Transforming Nonhomogeneous BCs into Homogeneous Ones

- Our goal is to find the **transient**  $U(x, t)$ . By substituting

$$u(x, t) = \left[ k_1 + \frac{x}{L}(k_2 - k_1) \right] + U(x, t)$$

in the original problem 6-1, we will arrive at a new problem in  $U(x, t)$ .

- We can solve this new problem for  $U(x, t)$  and add it to the steady state to get  $u(x, t)$ .

# Transforming Nonhomogeneous BCs into Homogeneous Ones

## Problem 6-1a

$$\text{PDE: } U_t = \alpha^2 U_{xx}, \quad 0 < x < L,$$

$$\text{BCs: } \begin{cases} U(0, t) = 0 \\ U(L, t) = 0 \end{cases}, \quad 0 < t < \infty$$

$$\text{IC: } U(x, 0) = \bar{\phi}(x), \quad 0 \leq x \leq L,$$

where  $\bar{\phi}(x) := \phi(x) - [k_1 + \frac{x}{L}(k_2 - k_1)]$  new IC. But it is known!!!

The problem 6-1a (fortunately) has a homogeneous PDE as well as homogeneous BCs, and so we can solve it by separation of variables.

### Question:

What about more realistic-type BCs with **time-varying** right-hand sides?

### Answer:

The ideas are similar to the previous problem 6-1 but a little more complicated.

# Transforming Time Varying BCs to Zero BCs

## Problem 6-2

Consider the typical problem

$$\text{PDE:} \quad u_t = \alpha^2 u_{xx}, \quad 0 < x < L, \quad 0 < t < \infty$$

$$\text{BCs:} \quad \begin{cases} u(0, t) = g_1(t) \\ u_x(L, t) + hu(L, t) = g_2(t) \end{cases}, \quad 0 < t < \infty$$

$$\text{IC:} \quad u(x, 0) = \phi(x), \quad 0 \leq x \leq L.$$



# Transforming Time Varying BCs to Zero BCs

- We seek a solution of the form

$$u(x, t) = A(t) \left[ 1 - \frac{x}{L} \right] + B(t) \frac{x}{L} + U(x, t)$$

where  $A(t)$  and  $B(t)$  are chosen so that the steady-state part

$$S(x, t) = A(t) \left[ 1 - \frac{x}{L} \right] + B(t) \frac{x}{L}$$

satisfies the BCs of the problem 6-2.

- The transformed problem in  $U(x, t)$  will have homogeneous BCs.

# Transforming Time Varying BCs to Zero BCs

- Substituting  $S(x, t)$  into BCs we get equations for  $A(t)$  and  $B(t)$

$$A(t) = g_1(t)$$

$$B(t) = \frac{g_1(t) + Lg_2(t)}{1 + Lh}.$$

- Hence, we have

$$u(x, t) = g_1(t) \left[ 1 - \frac{x}{L} \right] + \frac{g_1(t) + Lg_2(t)}{1 + Lh} \frac{x}{L} + U(x, t).$$

# Transforming Time Varying BCs to Zero BCs

The transformed problem in  $U(x, t)$  has a form

## Problem 6-2a

$$\text{PDE: } U_t = \alpha^2 U_{xx} - S_t, \quad (\text{nonhomogeneous PDE})$$

$$\text{BCs: } \begin{cases} U(0, t) = 0 \\ U_x(L, t) + hU(L, t) = 0 \end{cases}, \quad (\text{homogeneous BCs})$$

$$\text{IC: } U(x, 0) = \phi(x) - S(x, 0), \quad (\text{new IC - but known}).$$

The problem 6-2a has zero BCs (unfortunately, the PDE is nonhomogeneous). We can't solve this problem by separation of variables. But it can be solve by some other methods.

## Remark

For BCs of the form

$$\begin{cases} u(0, t) = g_1(t) \\ u(L, t) = g_2(t) \end{cases}$$

the method discussed in the problem 6-2 will give us the transformation

$$u(x, t) = g_1(t) + \frac{x}{L} [g_2(t) - g_1(t)] + U(x, t).$$

# More Complicated Problems and Separation of Variables

We start with a 1-dimensional heat-flow problem where one of the BCs contains derivatives.

## Heat-Flow Problem with Derivative BC

- Suppose we have a laterally insulated rod of length 1.
- Consider an apparatus in which we fix the temperature at the left end of the rod at  $u(0, t) = 0$  and immerse the right end of the rod in a solution of water fixed at the same temperature of zero (zero refers to some reference temperature).
- Newton's law of cooling says that the BC at  $x = 1$  is

$$u_x(1, t) = -hu(1, t).$$

- Suppose now that the initial temperature of the rod is  $u(x, 0) = x$ , but instantaneously thereafter ( $t > 0$ ), we apply our BCs.

To find the ensuing temperature, we must solve the IBVP

### Problem 6-3

To find the function  $u(x, t)$  that satisfies

$$\text{PDE:} \quad u_t = \alpha^2 u_{xx}, \quad 0 < x < 1, \quad 0 < t < \infty$$

$$\text{BCs:} \quad \begin{cases} u(0, t) = 0 \\ u_x(1, t) + hu(1, t) = 0 \end{cases}, \quad 0 < t < \infty$$

$$\text{IC:} \quad u(x, 0) = x, \quad 0 \leq x \leq 1$$

To apply the separation of variables method, we carry out the following steps:

## Step 1 (Finding elementary solutions to the PDE)

We look for solutions of the form  $u(x, t) = X(x)T(t)$  by substituting  $X(x)T(t)$  into the PDE and solving for  $X(x)T(t)$ . As a result we get

$$u(x, t) = e^{-\lambda^2 \alpha^2 t} (A \sin(\lambda x) + B \cos(\lambda x))$$

(with  $A$ ,  $B$  and  $\lambda$  arbitrary).



## Step 2 (Finding solutions to the PDE and the BCs)

- The next step is to choose a certain **subset** of solutions

$$e^{-\lambda^2 \alpha^2 t} (A \sin(\lambda x) + B \cos(\lambda x)) \quad (6.1)$$

that satisfy BCs.

- To do this, we substitute solutions (6.1) into BCs, getting

$$B e^{-\lambda^2 \alpha^2 t} = 0 \quad \Rightarrow \quad B = 0$$

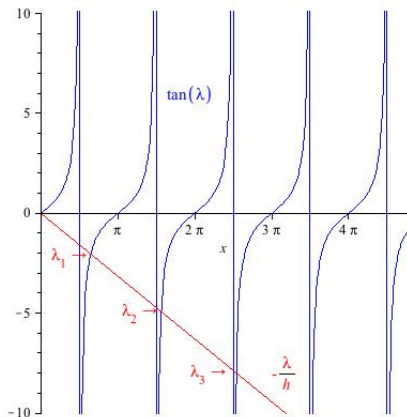
$$A \lambda e^{-\lambda^2 \alpha^2 t} \cos \lambda + h A e^{-\lambda^2 \alpha^2 t} \sin \lambda = 0.$$

Performing a little algebra on this last equation gives us the condition on  $\lambda$

$$\tan(\lambda) = -\lambda/h.$$

## Step 2 (Finding solutions to the PDE and the BCs)

In other words, to find  $\lambda$ , we must find the intersections of the curves  $\tan(\lambda)$  and  $-\lambda/h$ .



## Step 2 (Finding solutions to the PDE and the BCs)

- These values  $\lambda_1, \lambda_2, \dots$  can be computed numerically for a given  $h$  on a computer and are called the **eigenvalues** of the boundary-value problem

$$\begin{aligned}X'' + \lambda^2 X &= 0 \\X(0) &= 0 \\X'(1) + hX(1) &= 0\end{aligned}\tag{6.2}$$

In other words, they are the values of  $\lambda$  for which there exist a **nonzero solution**.

## Step 2 (Finding solutions to the PDE and the BCs)

The eigenvalues  $\lambda_n$  of (6.2), which, in this case, are the roots of

$$\tan(\lambda) = -\lambda/h,$$

have been computed (for  $h = 1$ ) numerically, and the first five values are listed in Table 6.1

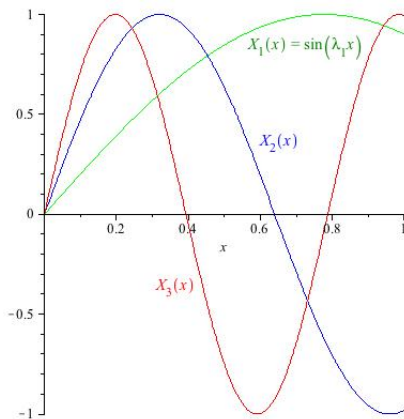
$n$	$\lambda_n$
1	2,02
2	4,91
3	7,98
4	11,08
5	14,20

Table 6.1: Roots of  $\tan(\lambda) = -\lambda$ .

## Step 2 (Finding solutions to the PDE and the BCs)

- The solutions of (6.2) corresponding to the eigenvalues  $\lambda_n$  are called the **eigenfunctions**  $X_n(x)$ , and for problem (6.2), we have

$$X_n(x) = \sin(\lambda_n x).$$



## Step 2 (Finding solutions to the PDE and the BCs)

- We have now finished the second step; we have found an infinite number of functions (fundamental solutions),

$$u_n(x, t) = e^{-\lambda_n^2 \alpha^2 t} \sin(\lambda_n x)$$

each one satisfying the PDE and BCs.

## Step 3 (Finding solutions to the PDE, BCs, and the IC)

- The last step is to add the fundamental solutions

$$u(x, t) = \sum_{n=1}^{\infty} A_n e^{-(\lambda_n \alpha)^2 t} \sin(\lambda_n x) \quad (6.3)$$

in such a way (pick the coefficients  $A_n$ ) that the initial condition  $u(x, 0) = x$  is satisfied.

- Substituting (6.3) into the IC gives

$$x = \sum_{n=1}^{\infty} A_n \sin(\lambda_n x).$$

## Step 3 (Finding solutions to the PDE, BCs, and the IC)

We are now in position to solve for the coefficients in the expression

$$x = \sum_{n=1}^{\infty} A_n \sin(\lambda_n x). \quad (6.4)$$

- We **multiply** each side of (6.4) by  $\sin(\lambda_m x)$  ( $m$  is an arbitrary integer) and **integrate** from zero to one. As a result we get

$$\begin{aligned} \int_0^1 x \sin(\lambda_m x) dx &= A_m \int_0^1 \sin^2(\lambda_m x) dx \\ &= A_m \left( \frac{\lambda_m - \sin(\lambda_m) \cos(\lambda_m)}{2\lambda_m} \right). \end{aligned}$$



## Step 3 (Finding solutions to the PDE, BCs, and the IC)

- Solving for  $A_n$  (we'll change the notation to  $A_n$ ), we get

$$A_n = \frac{2\lambda_n}{\lambda_n - \sin(\lambda_n) \cos(\lambda_n)} \int_0^1 x \sin(\lambda_n x) dx. \quad (6.5)$$

- We are done; our solution is

$$u(x, t) = \sum_{n=1}^{\infty} A_n e^{-(\lambda_n \alpha)^2 t} \sin(\lambda_n x),$$

where the constants  $A_n$  are given by (6.5).

For problem 6-3 the first five constants  $A_n$  have been computed and are listed in Table 6.2:

$n$	$A_n$
1	0,24
2	0,22
3	-0,03
4	-0,11
5	-0,09

Table 6.2: Coefficients  $A_n$ .

Hence, the first three terms of the IBVP 6-3 are

$$u(x, t) \approx 0,24e^{-4t} \sin(2x) + 0,22e^{-24t} \sin(4,9x) \\ + 0,03e^{-63,3t} \sin(7,98x) + \dots$$

## Remark

- The eigenvalue problem (6.2) is a special case of the general problem

$$\text{ODE: } [p(x)Y'(x)]' - q(x)Y(x) + \lambda r(x)Y(x) = 0,$$

$$\text{BCs: } \begin{cases} \alpha_1 Y(0) + \beta_1 Y'(0) = 0 \\ \alpha_2 Y(1) + \beta_2 Y'(1) = 0 \end{cases},$$

known as **Sturm-Liouville problem**.