Universität des Saarlandes
Fachrichtung 6.1 – Mathematik
PD Dr. Darya Apushkinskaya

UNIVERSITÄT DES SAARLANDES

PDE and Boundary-Value Problems (Winter Term 2016/2017)
Exam-Part W

Problem E.1 (10 Points)

Solve the problem

$$\begin{array}{lll} \text{PDE} & u_t = u_{xx} - u + x, & 0 < x < 1, \quad 0 < t < \infty \\ \text{BCs} & \begin{cases} u(0,t) = 1 \\ u(1,t) = 0 \end{cases} & 0 < t < \infty \\ \text{IC} & u(x,0) = 0 & 0 \leqslant x \leqslant 1 \end{array}$$

by

(a) changing the nonhomogeneous BCs to homogeneous ones.

(b) transforming into a new equation without the term $-u$.

(c) solving the resulting problem.

Problem E.2 (10 Points)

We seek the electrostatic potential in a charge-free rectangular domain

$$D = \left\{(x, y) \in \mathbb{R}^2 : 0 < x < 1, 0 < y < 1\right\}.$$

The sides $x = 0$, $x = 1$, and $y = 1$ are held at a fixed potential zero, and the side $y = 0$ has a potential distribution $u(x, 0) = x(1 - x)$.

Using Maple construct the three-dimensional surface which depicts the electrostatic potential distribution $u(x, y)$ over the rectangular region. Note how the edges of the surface adhere to the given boundary conditions. The equipotential lines can be obtained from Maple by clicking on the the figure and the choosing the special option "Render the plot using the polygon patch and contour style" in the graphics bar and then clicking the "redraw" button.

Problem E.3 (10 Points)

(a) Find the finite-difference solution of the heat-conduction problem

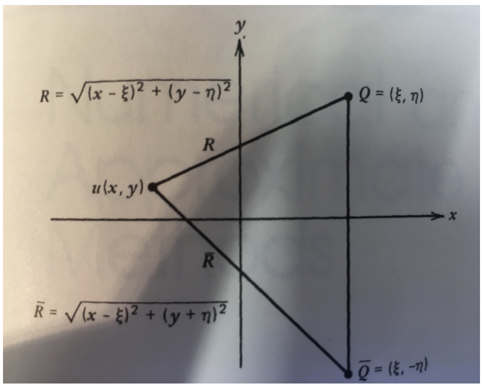
$$
\begin{array}{lll}
\text{PDE} & u_t = u_{xx}, & 0 < x < 1, \quad 0 < t < \infty \\
\text{BCs} & \begin{cases} u(0,t) = 0 \\ u_x(1,t) = 1 - u(1,t) \end{cases} & 0 < t < \infty \\
\text{IC} & u(x,0) = sin(\pi x) & 0 \leqslant x \leqslant 1
\end{array}
$$

for $t = 0.005,\ 0.010,\ 0.015$ by the explicit method. Let $h = \Delta x = 0.1$. Plot the solution at $x = 0,\ 0.1,\ 0.2,\ 0.3,\ \ldots,\ 0.9,\ 1$ for $t = 0.015$.

(b) Solve the problem form part (a) analytically (separation of variables) and evaluate the analytical solution at the grid points: $x = 0,\ 0.1,\ 0.2,\ 0.3,\ \ldots,\ 0.9,\ 1$ for $t = 0.015$. Compare these results to your numerical solution. (You may wish to write a small computer program or use a calculator to evaluate the separation-of-variables solution).

Problem E.4 (10+4=14 Points)

(a) Let $n = 2$. Find Green's function $G(x, y, \xi, \eta)$ for the Laplace equation in the upper-half plane $y > 0$. In other words, find the potential in the upper-half plane at the point (x, y) (zero on the boundary $y = 0$) due to a point charge at (ξ, η). See the following figure:



HINT: If we place a negative charge at $\overline{Q} = (\xi, -\eta)$, then it's clear that the potential field on the line $y = 0$ due to two charges at Q and $\overline{Q}$ is zero. Hence, Green's function would be the resultant field due to these two charges.

(b) Using the results of problem (a), what is the solution to Poisson's equation

$$\Delta u = -k$$

in the upper-half plane with zero BC?

Problem E.5 (10 Points)

Derive the Monte-Carlo game for solwing the following parabolic IBVP:

$$
\begin{array}{rll}
\text{PDE} & u_t = \alpha^2 u_{xx} - \beta u_x - \gamma u, & 0 < x < 1,\ t > 0, \\
\text{BCs} & \begin{cases} u(0,t) = f(t) \\ u(1,t) = g(t) \end{cases} & t > 0. \\
\text{IC} & u(x,0) = \phi(x) & 0 \leqslant x \leqslant 1
\end{array}
$$

HINT: Replace the PDE by the finite-difference approximation from the Crank-Nicolson method; solve for $u_{i+1,j}$ in terms of its five neighbors $u_{i+1,j-1}$, $u_{i+1,j+1}$, $u_{i,j-1}$, $u_{i,j}$, $u_{i,j+1}$, and go on from there.

Deadline for submission: Tuesday, April 18