

# PDE and Boundary-Value Problems

## Winter Term 2016/2017

### Lecture 1

Universität des Saarlandes

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<http://www.num.uni-sb.de/rjasanow/dokuwiki/doku.php?id=ag:apushkinskaya>

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- More details on the web page:

<http://www.num.uni-sb.de/rjasanow/dokuwiki/doku.php?id=lehre:vorlesung:pdebvp1617>



# Registration

Registration for the lecture course PDEs and BVPs 2016-2017

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## Registration form for the lecture course PDEs and BVPs 2016-2017

You have to fill everything.

Name:	<input type="text"/>
Family name:	<input type="text"/>
Matrikelnummer:	<input type="text"/>
Date of birth (TT.MM.YYYY):	<input type="text"/>
Email:	<input type="text"/>
Email (Wdh):	<input type="text"/>
Study major :	<input type="text" value="--"/>
40+10:	<input type="text"/>
	<input type="button" value="Submit"/>

[Impressum](#)



## What type of Lectures is it?

- Lectures (3h) with exercises (1h) (6 ECTS points)
- Time and Location: **Wednesday 10-12 c.t.**, and Building E1.3, Lecture Room 003 **Friday 08:30-10 c.t.**, Building E1.3, Lecture Room 001
- Exercises: Every second Friday instead of a lecture
- First exercise: Friday, November 4, 2016



## What Prerequisites are required?

- Undergraduate knowledge in mathematics (i.e., Calculus I and II, Linear Algebra)
- Passive knowledge of simple English. (Exam can be also submitted in German or in Russian)
- Basic knowledge of Maple



## Assignments:

- Homework will be assigned bi-weekly.
- To qualify for the exam you need 50% of the points from these assignments.
- Working in groups of up to 2 people is permitted



## Exams:

There will be an written-oral exam.



## Contents:

- 1 Introduction to partial differential equations,
- 2 Parabolic-type problems,
- 3 Hyperbolic-type problems,
- 4 Elliptic-type problems





## Script:

Course material would be available on the webpage in order to **support** the classroom teaching, **not to replace** it.

Additional organisational information, examples and explanations that may be relevant for your understanding and the exam are provided in the lectures.

It is solely **your** responsibility to make sure that you receive this information.



## Literature:



L.C. Evans,  
*Partial Differential Equations*  
Amer. Math. Soc., Providence, RI (1998).



M.A. Pinsky,  
*Partial Differential Equations and Boundary-Value Problems with Applications.*  
Amer. Math. Soc., Providence, RI, 2011



S.J. Farlow, , Dover Publications, INC. New York, 1993.  
*Partial Differential Equations for Scientists and Engineers.*  
Dover Publications, INC. New York, 1993.



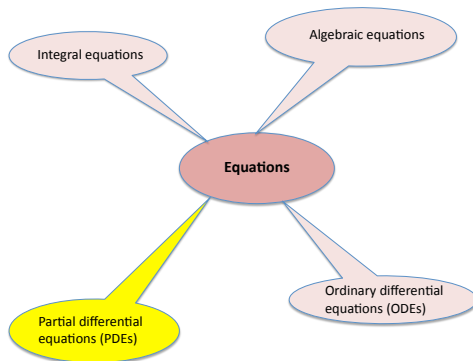
## Chapter 1. Introduction to Partial Differential Equations

### Purpose of Lesson:

- To show what PDEs are, why they are useful, and how they are solved.
- A brief discussion on how PDEs are classified as various kinds and types.



# What are PDEs?



- **Algebraic equations** state relations between **unknown number** and **its power**,

e.g.

$$x^3 - 7x^2 - 44x = 0.$$

Three solutions:  $x_1 = 0$ ,  $x_2 = -4$ ,  $x_3 = 11$ .

- **Ordinary differential equations** state relations between an **unknown function** of **ONE variable** and **its derivatives**,

e.g.

$$u''(t) = 0.$$

Infinitely many solution:  $u(t) = at + b$  ( $a, b \in \mathbb{R}$ ).



- Partial differential equations state relations between an unknown function of SEVERAL variables and its partial derivatives, e.g.

$$u_t(x, t) = u_{xx}(x, t) \quad (\text{heat equation})$$

$$u_t(x, y, t) = u_{xx}(x, y, t) + u_{yy}(x, y, t) \quad (\text{heat equation})$$

$$u_{tt}(x, y, z, t) = u_{xx} + u_{yy} + u_{zz} \quad (\text{wave equation})$$

$$u_{tt}(x, t) = u_{xx} + \alpha u_t + \beta u \quad (\text{telegraph equation})$$



# Examples of PDEs:

$$\Delta u = \sum_{i=1}^n u_{x_i x_i} = 0$$

Laplace's equation

$$u_t - \Delta u = 0$$

heat (or diffusion) equation

$$u_{tt} - \Delta u = 0$$

wave equation

$$u_t - \Delta u - \sum_{i=1}^n (b^i u)_{x_i} = 0$$

Fokker-Planck equation

$$\operatorname{div} \left( \frac{Du}{\sqrt{1+|Du|^2}} \right) = 0$$

minimal surface equation

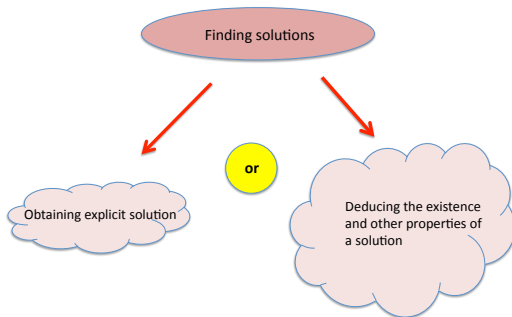


How do you solve a PDE?

We solve the PDE if we find all functions verifying our PDE.







## How do you solve a PDE?

The most important methods are those that change PDEs into ODEs. The useful techniques are:

- *Separation of Variables*. This technique reduce a PDE in  $n$  variables to  $n$  ODEs.
- *Integral Transforms*. This procedure reduces a PDE in  $n$  independent variables to one in  $n - 1$  variables; hence, a PDE in two variables could be changed to an ODE.
- *Eigenfunction Expansion*. This method attempts to find the solution of a PDE as an infinite sum of *eigenfunctions*. These eigenfunctions are found by solving what is known as an eigenvalue problem.
- *Numerical Methods*.



## Remarks.

- There is no general theory known concerning the solvability of **all** PDEs.
- Such a theory is extremely unlikely to exist, given a rich variety of physical, geometric and probabilistic phenomena which can be modelled by PDEs.
- Instead, research focuses on various particular PDEs that are important for applications.

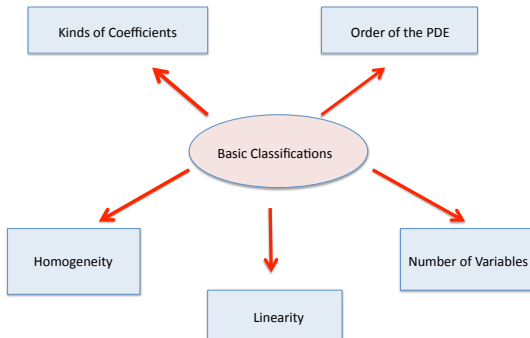


# Kinds of PDEs.

PDE are classified according many things.

Classification is an important concept because the general theory and methods usually apply only to a given class of equations.





The basic classifications are:

- *Order of PDE*. The order of a PDE is the order of the **highest partial derivative** in the equation.
- *Number of Variables*. The number of variables is the **number of independent** variables.
- *Linearity*. PDEs are either **linear** or **nonlinear**. A PDE is said to be linear if it can be written in the form

$$\mathcal{L}u = g,$$

where  $g$  is a given function and  $\mathcal{L}$  is a linear differential operator, i.e.,

$$\mathcal{L}(u + v) = \mathcal{L}u + \mathcal{L}v, \quad \mathcal{L}(cu) = c\mathcal{L}u, \quad c \in \mathbb{R}.$$



- *Homogeneity*. A linear PDE is said to be **homogeneous** if it can be written in the form

$$\mathcal{L}u = g,$$

where  $g$  is a given function and  $\mathcal{L}$  is a linear differential operator, and  $g \equiv 0$ . If a function  $g$  is not identically zero, then our linear PDE is called **nonhomogeneous**.

- *Kind of Coefficients*. If the coefficients of differential operator  $\mathcal{L}$  are constants, then equation  $\mathcal{L}u = g$  is said to have **constant coefficients** (otherwise, **variable coefficients**).



## Remarks.

- PDE theory is (mostly) not restricted to two independent variables.
- Many interesting equations are nonlinear.





First we classify the 2nd order PDE in two independent variables which is linear with respect to its second-order derivatives:

$$a(x, y)u_{xx} + b(x, y)u_{xy} + c(x, y)u_{yy} + f(x, y, u, u_x, u_y) = 0. \quad (1.1)$$

Here  $a, b, c, f$  are given differentiable functions.

- If  $b^2 - 4ac < 0$ , then PDE (1.1) is called **elliptic**.
- If  $b^2 - 4ac = 0$ , then PDE (1.1) is called **parabolic**.
- If  $b^2 - 4ac > 0$ , then PDE (1.1) is called **hyperbolic**.



## Remark

All PDEs like (1.1) (in 2 independent variables!!!) are either

- parabolic
- elliptic
- hyperbolic.

## Remark

In three and more dimensions, 2nd order PDEs can be one type in one pair of variables and of another type in other variables, e.g., there can occur elliptic-hyperbolic equations, ultra-hyperbolic equations, etc.



## Remark

The nature of PDE depends only on the coefficients of the second order terms. First order terms and zero order terms do not play a role here.

## Remark

The type of the 2nd order PDE can be different in different regions.



- Parabolic equations describe heat flow and diffusion processes.
- Hyperbolic equations describe vibrating systems and wave motion.
- Elliptic equations describe steady-state phenomena.



# Normal forms of 2nd order PDEs in two independent variables:

Using a suitable transformation of independent variables

$$\xi = \xi(x, y), \quad \eta = \eta(x, y)$$

we can always reduce equation

$$a(x, y)u_{xx} + b(x, y)u_{xy} + c(x, y)u_{yy} + f(x, y, u, u_x, u_y) = 0$$

to one of the following three NORMAL FORMs:



- for **hyperbolic** equations

$$u_{\xi\eta} = F(\xi, \eta, u, u_{\xi}, u_{\eta}), \quad \text{or} \quad u_{\xi\xi} - u_{\eta\eta} = F(\xi, \eta, u, u_{\xi}, u_{\eta});$$

- for **parabolic** equations

$$u_{\eta\eta} = F(\xi, \eta, u, u_{\xi}, u_{\eta}),$$

where  $F$  **must** depend on  $u_{\xi}$ : otherwise the equation degenerates into an ODE;

- for **elliptic** equations

$$u_{\xi\xi} + u_{\eta\eta} = F(\xi, \eta, u, u_{\xi}, u_{\eta}).$$



The classification (elliptic, parabolic etc.) can be extended to equations depending on more than 2 variables.

Consider the 2nd order PDE depending on  $n$  variables,

$$\sum_{i=1}^n \sum_{j=1}^n a_{ij} u_{x_i x_j} + \sum_{i=1}^n b_i u_{x_i} + cu + g = 0. \quad (1.2)$$

The coefficient matrix  $(a_{ij})$  should be symmetrized because

$$\frac{\partial^2}{\partial x_i \partial x_j} = \frac{\partial^2}{\partial x_j \partial x_i}, \quad \text{for any } i \text{ and } j \text{ in } [1, n].$$



The classification is as follows:

- **hyperbolic** for  $(Z = 0 \text{ and } P = 1)$  or  $(Z = 0 \text{ and } P = n - 1)$
- **parabolic** for  $Z > 0$  ( $\Leftrightarrow \det(a_{ij}) = 0$ )
- **elliptic** for  $(Z = 0 \text{ and } P = n)$  or  $(Z = 0 \text{ and } P = 0)$
- **ultra-hyperbolic** for  $(Z = 0 \text{ and } 1 < P < n - 1)$

where

$Z$  = number of **zero** eigenvalues  $(a_{ij})$ ,

$P$  = number of **strictly positive** eigenvalues of  $(a_{ij})$ .

