

PDE and Boundary-Value Problems

Winter Term 2016/2017

Lecture 11

Saarland University

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Purpose of Lesson

- To derive the **fundamental solution** of the heat equation and discuss the corresponding solutions of homogeneous and nonhomogeneous IVPs.
- To discuss some properties of solutions to the heat equations.

Fundamental solution of the heat equation:

Problem 11-1

To find the function $u(x, t)$ that satisfies

$$\text{PDE: } u_t = \Delta u, \quad x \in \mathbb{R}^n, \quad 0 < t < \infty$$

$$\text{IC: } u(x, 0) = u_0, \quad x \in \mathbb{R}^n$$

We will solve problem 11-1 by applying the exponential Fourier transform with respect to the spatial variables x .

We define

$$U(t, \xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} u(t, x) e^{-i\xi \cdot x} dx, \quad \xi \in \mathbb{R}^n.$$

Step 1. (Transformation)

- We take n Fourier transforms with respect to all x_j -variables. As a result we get the following ODE in t

$$\begin{aligned} \text{ODE: } U_t(t) + |\xi|^2 U(t) &= 0, & 0 < t < \infty \\ \text{IC: } U(0) &= \mathcal{F}[u_0] \end{aligned} \tag{11.1}$$

Step 2. (Solving the transformed problem)

- Remember the new variable ξ is nothing more than a constant vector in this differential equation. So, the solution to problem (11.1) is

$$U(t, \xi) = \mathcal{F}[u_0](\xi) e^{-|\xi|^2 \cdot t}.$$

Step 3. (Finding the inverse transform)

- We compute

$$u(x, t) = \mathcal{F}^{-1} [U(t, \xi)] = \mathcal{F}^{-1} \left[\mathcal{F}[u_0](\xi) e^{-|\xi|^2 \cdot t} \right]$$

- Due to the convolution property we can write

$$\begin{aligned} u(x, t) &= \mathcal{F}^{-1} [\mathcal{F}[u_0](\xi)] * \mathcal{F}^{-1} \left[e^{-|\xi|^2 \cdot t} \right] \\ &= u_0(x) * \left[\frac{1}{(2t)^{n/2}} e^{-|x|^2/(4t)} \right] \\ &= \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-|x-y|^2/(4t)} u_0(y) dy \end{aligned}$$

Fundamental solution

The function

$$\Phi(x, t) := \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/(4t)}, & x \in \mathbb{R}^n, t > 0 \\ 0, & x \in \mathbb{R}^n, t < 0 \end{cases}$$

is called the **fundamental solution** of the heat equation.

The fundamental solution has the following properties:

- Φ is singular at the point $(0, 0)$
- $\Phi(x, t) = \Phi(|x|, t)$, i.e., the fundamental solution is radial in the variable x .
- For each time $t > 0$ we have $\int_{\mathbb{R}^n} \Phi(x, t) dx = 1$.
- $\Phi_t = \Delta \Phi$, $x \in \mathbb{R}^n$, $t > 0$.

Remarks

- If u_0 is bounded, continuous, $u_0 \geq 0$, and $u_0 \not\equiv 0$, then

$$u(x, t) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-|x-y|^2/(4t)} u_0(y) dy$$

is in fact positive for **all** points $x \in \mathbb{R}^n$ and times $t > 0$.

- We interpret the above observation by saying the heat equation forces **infinite propagation speed** for disturbances.
- If the initial temperature is nonnegative and is positive somewhere, the temperature at any later time is everywhere positive.

Now let us turn our attention to the **nonhomogeneous** IVP.

Problem 11-2

To find the function $u(x, t)$ that satisfies

$$\text{PDE: } u_t - \Delta u = f, \quad x \in \mathbb{R}^n, \quad 0 < t < \infty$$

$$\text{IC: } u(x, 0) = 0, \quad x \in \mathbb{R}^n$$

Solving of problem 11-2:

- First of all, we recall that the mapping

$$(x, t) \mapsto \Phi(x - y, t - s)$$

is a solution of the heat equation (for given $y \in \mathbb{R}^n$, $0 < s < t$).

- Now for fixed s , the function

$$u = u(x, t, s) = \int_{\mathbb{R}^n} \Phi(x - y, t - s) f(y, s) dy$$

solves the problem

$$\begin{cases} u_t(\cdot, s) - \Delta u(\cdot, s) = 0 & \text{in } \mathbb{R}^n \times (s, \infty) \\ u(\cdot, s) = f(\cdot, s) & \text{on } \mathbb{R}^n \times \{t = s\}. \end{cases} \quad (11.1s)$$

Solving of problem 11-2 (cont.):

- Observe that (11.1s) is an IVP of the form 11-1, with the starting time $t = 0$ replaced by $t = s$, and u_0 replaced by $f(\cdot, s)$.
- Duhamel's principle asserts that we can build a solution of problem 11-2 out of the solutions of (11.1s), by integrating with respect to s . The idea is to consider

$$u(x, t) = \int_0^t u(x, t, s) ds.$$

Solving of problem 11-2 (cont.):

- Rewriting, we have

$$\begin{aligned}u(x, t) &= \int_0^t \int_{\mathbb{R}^n} \Phi(x - y, t - s) f(y, s) dy ds \\ &= \int_0^t \frac{1}{(4\pi(t - s))^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4(t-s)}} f(y, s) dy ds,\end{aligned}$$

for $x \in \mathbb{R}^n$, $t > 0$.

Combining the solutions of problems 11-1 and 11-2, we discover that

$$u(x, t) = \int_{\mathbb{R}^n} \Phi(x - y, t) u_0(y) dy + \int_0^t \int_{\mathbb{R}^n} \Phi(x - y, t - s) f(y, s) ds$$

is a solution of the nonhomogeneous problem

Problem 11-3

$$\text{PDE: } u_t - \Delta u = f, \quad x \in \mathbb{R}^n, \quad 0 < t < \infty$$

$$\text{IC: } u(x, 0) = u_0, \quad x \in \mathbb{R}^n$$

Assume $U \subset \mathbb{R}^n$ is open and bounded, and fix a time $T > 0$.

Remarks

- We define the **parabolic cylinder**

$$U_T := U \times (0, T].$$

- The **parabolic boundary** of U_T is

$$\partial' U_T := \overline{U_T} \setminus U_T.$$

- We note that U_T includes the top $U \times \{t = T\}$. The parabolic boundary $\partial' U_T$ comprises the bottom and vertical sides of $U \times [0, T]$, but not the top.

Properties of solutions to the heat equation

1. (Strong maximum principle)

Assume $u \in C^{2,1}(U_T) \cap C(\overline{U_T})$ solves the heat equation in the parabolic cylinder U_T .

(i) Then

$$\max_{\overline{U_T}} u = \max_{\partial' U_T} u.$$

(ii) Furthermore, if U is connected and there exists a point $(x_0, t_0) \in U_T$ such that

$$u(x_0, t_0) = \max_{\overline{U_T}} u,$$

then

u is a constant in $\overline{U_{T_0}}$.

Remarks

- Assertion (i) is the **maximum principle** for the heat equation and (ii) is the **strong maximum principle**.
- Similar assertions are valid with „min“ replacing „max“.
- So if u attains its maximum (or minimum) at the interior point, then u is constant at all earlier times. The solution may change at times $t > t_0$, provided the boundary conditions alter after t_0 .

Properties of solutions to the heat equation (cont.)

2. (Uniqueness on bounded domains)

Let $g \in C(\partial' U_T)$ and $f \in C(U_T)$. Then there exists **at most one** solution

$$u \in C^{2,1}(U_T) \cap C(\overline{U_T})$$

of the problem

$$\begin{cases} u_t - \Delta u = f & \text{in } U_T \\ u = g & \text{on } \partial' U_T. \end{cases}$$

Properties of solutions to the heat equation (cont.)

3. (Smoothness)

Suppose $u \in C^{2,1}(U_T)$ solves the heat equation in U_T . Then

$$u \in C^\infty(U_T).$$

Remark

- The regularity assertion is valid even if u attains nonsmooth boundary values on $\partial' U_T$.