# PDE and Boundary-Value Problems
# Winter Term 2016/2017

Lecture 12

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## Purpose of Lesson

- To introduce the one-dimensional wave equation and show how it describes the motion of a vibrating string.
- To show how the one-dimensional wave equation is derived as a result of Newton's equations of motion.
- To find D'Alembert solution of the wave equation and interpretate it in terms of moving wave motion.
- To interpretate the D'Alembert solution in the *xt*-plane.

## Chapter 3. Hyperbolic-Type Problems

So far, we have been concerned with physical phenomenon described by parabolic equations. We will now begin to study the second major class pf PDEs, hyperbolic equations.

We start by studying the one-dimensional wave equation, which describes (among other things) the transverse vibrations of a string.

# Vibrating-String Problem

Suppose we have the following simple experiment that we break into steps.

1. Consider the small vibrations of a string length $L$ that is fastened at each end.
2. We assume the string is stretched tightly, made of a homogeneous material, unaffected by gravity, and that the vibrations take place in a plane.

# The mathematical model of the vibrating-string problem

To mathematically describe the vibrations of the 1-dimensional string, we consider all the forces acting on a small section of the string.

Essentially, the wave equation is nothing more than Newton's equation of motion applied to the string (the change of momentum $mu_{tt}$ of a small string segment is equal to the applied forces).

The most important forces are

1. Net force due to the tension of the string $(\alpha^2 u_{xx})$

   The tension component has a net transverse force on the string segment of

$$\begin{aligned}\text{Tension component} &= T\sin(\theta_2) - T\sin(\theta_1)\\ &\approx T\left[u_x(x+\Delta x, t) - u_x(x,t)\right]\end{aligned}$$

2. External force $F(x,t)$

   An external force $F(x,t)$ may be applied along the string at any value of $x$ and $t$.

3. Frictional force against the string $(-\beta u_t)$

   If the string is vibrating in a medium that offers a resistance to the string's velocity $u_t$, then this resistance force is $-\beta u_t$.

4. Restoring force $(-\gamma u)$

   This is a force that is directed opposite to the displacement of the string. If the displacement $u$ is positive (above the $x$-axis), then the force is negative (downward).

If we now apply Newton's equation of motion

$$mu_{tt} = \text{applied forces to the segment } (x, x + \Delta x)$$

to the small segment of string, we have

$$\Delta x \rho u_{tt}(x,t) = T\left[u_x(x+\Delta x, t) - u_x(x,t)\right] + \Delta x F(x,t) \\ - \Delta x \beta u_t(x,t) - \Delta x \gamma u(x,t),$$

where $\rho$ is the density of the string.

By dividing each side of the equation by $\Delta x$ and letting $\Delta x \to 0$, we have the equation

$$u_{tt} = \alpha^2 u_{xx} - \delta u_t - \kappa u + f(x,t),$$

where $\alpha^2 = \dfrac{T}{\rho}$, $\delta = \dfrac{\beta}{\rho}$, $\kappa = \dfrac{\gamma}{\rho}$, and $f(x,t) = \dfrac{F(x,t)}{\rho}$.

# Intuitive Interpretation of the Wave Equation

- The expression $u_{tt}$ represents the vertical acceleration of the string at a point $x$.
- Equation

$$u_{tt} = \alpha^2 u_{xx}$$

can be interpreted as saying that the acceleration of each point of the string is due to the tension in the string and that the larger the concavity $u_{xx}$, the stronger the force.

## Remarks

- If the vibrating string had a variable density $\rho(x)$, then the wave equation would be
$$u_{tt} = \frac{\partial}{\partial x}\left[\alpha^2(x)u_x\right].$$
In other words, the PDE would have variable coefficients.

## Remarks (cont.)

- Since the wave equation $u_{tt} = \alpha^2 u_{xx}$ contains a second-order time derivative $u_{tt}$, it requires two initial conditions

$$u(x,0) = f(x) \qquad \text{(initial position of the string)}$$
$$u_t(x,0) = g(x) \qquad \text{(initial velocity of the string)}$$

in order to uniquely define the solution for $t > 0$. This is in contrast to the heat equation, where only one IC was required.

# The D'Alembert Solution of the Wave Equation

- In the parabolic case we started solving problems when the space variable was bounded (by separation of variables) and then went on to solve the unbounded case (where $-\infty < x < \infty$) by the Fourier transform.
- In the hyperbolic case (wave problem), we will do the opposite.
- We start by solving the one-dimensional wave equation in free spece. We will use the method similar to the moving-coordinate method from diffusion-comvection equation.

## Problem 12-1

To find the function $u(x,t)$ that satisfies

$$\text{PDE:} \quad u_{tt} = c^2 u_{xx}, \qquad -\infty < x < \infty, \quad 0 < t < \infty$$

$$\text{ICs:} \quad \begin{cases} u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases} \qquad -\infty < x < \infty$$

We solve problem 12-1 by breaking it into several steps.

Step 1. (Replacing $(x, t)$ by new canonical coordinates $(\xi, \eta)$)

- We introduce two new space-time coordinates $(\xi, \eta)$

$$\xi = x + ct$$
$$\eta = x - ct$$

- In new variables our PDE takes the form

$$u_{\xi\eta} = 0. \tag{12.1}$$

## Step 2. (Solving the transformed equation)

- We solve (12.1) by two straightforward integrations (first with respect to $\xi$ and then with respect to $\eta$). The general solution of (12.1) is

$$u(\xi, \eta) = \phi(\eta) + \psi(\xi), \tag{12.2}$$

where $\phi(\eta)$ and $\psi(\xi)$ are arbitrary functions of $\eta$ and $\xi$, respectively.

## Step 3. (Transforming back to the original coordinates $x$ and $t$)

- We substitute

$$\xi = x + ct$$
$$\eta = x - ct$$

into (12.2) to get

$$u(x,t) = \phi(x - ct) + \psi(x + ct). \tag{12.3}$$

**Remark**

(12.3) is physically represents the sum of any two moving waves, each moving in opposite directions with velocity $c$.

## Step 4. (Substituting the general solution into the ICs)

- Substituting (12.3) into our ICs, we get

$$\begin{aligned} \phi(x) + \psi(x) &= f(x) \\ -c\phi'(x) + c\psi'(x) &= g(x) \end{aligned} \tag{12.4}$$

- Integrating the second equation of (12.4) from $x_0$ to $x$, we obtain

$$-c\phi(x) + c\psi(x) = \int_{x_0}^{x} g(s)ds + K. \tag{12.5}$$

## Step 4. (Substituting the general solution into the ICs (cont.))

- If we solve algebraically for $\phi(x)$ and $\psi(x)$ from the first equation of (12.4) and (12.5), we have

$$\phi(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int\limits_{x_0}^{x} g(s)ds - \frac{K}{2c}$$

$$\psi(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int\limits_{x_0}^{x} g(s)ds + \frac{K}{2c}$$

Step 4. (Substituting the general solution into the ICs (cont.))

- Hence, the solution to our problem 12-1 is

$$\boxed{u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int\limits_{x-ct}^{x+ct} g(s)ds}.$$

It is called the D'Alembert solution.

# Examples of the D'Alembert Solution

## 1. Motion of an Initial Sine Wave

- Consider the initial conditions

$$u(x,0) = \sin(x)$$
$$u_t(x,0) = 0$$

The initial sine wave would have the solution

$$u(x,t) = \frac{1}{2}\left[\sin(x-ct) + \sin(x+ct)\right]$$

- This can be interpreted as dividing the initial shape $u(x,0) = \sin(x)$ into two equal parts

$$\frac{\sin(x)}{2} \quad \text{and} \quad \frac{\sin(x)}{2}$$

and then adding the two resultant waves as one moves to the left and the other to the right (each with velocity $c$).

# Examples of the D'Alembert Solution (cont.)

2. Motion of a Simple Square Wave
   - In this case, if we start the initial conditions

$$u(x,0) = \begin{cases} 1, & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$
$$u_t(x,0) = 0$$

then the initial wave is decomposed into two half waves travelling in opposite direction.

# Examples of the D'Alembert Solution (cont.)

3. Initial Velocity Given

- Suppose now the initial position of the string is at equilibrium and we impose an initial velocity (as in piano string) of $\sin(x)$

$$u(x,0) = 0$$
$$u_t(x,0) = \sin(x)$$

- Here, the solution would be

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \sin(s)ds$$
$$= \frac{1}{2c}\left[\cos(x+ct) - \cos(x-ct)\right]$$

which represents the sum of two moving cosine wave.

## Remarks

- Note that a second-order PDE has two arbitrary functions in its general solution, whereas the general solution of a second-order ODE has two arbitrary constants. In other words, there are more solutions to a PDE than to an ODE.

- The general technique of changing coordinate systems in a PDE in order to find a simpler equation is common in PDE theory.

- The new coordinates $(\xi, \eta)$ in problem 12-1 are known as canonical coordinates.

- The strategy of finding the general solution to a PDE and then substituting it into the boundary and initial conditions is not a common technique in solving PDEs.

# The Space-Time Interpretation of D'Alembert's Solution

We present an interpretation of the D'Alembert solution

$$u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int\limits_{x-ct}^{x+ct} g(s)ds$$

in the *xt*-plane looking at two specific cases.

# Case 1. (Initial position given; initial velocity zero)

- Suppose the string has initial conditions

$$u(x,0) = f(x)$$
$$u_t(x,0) = 0$$

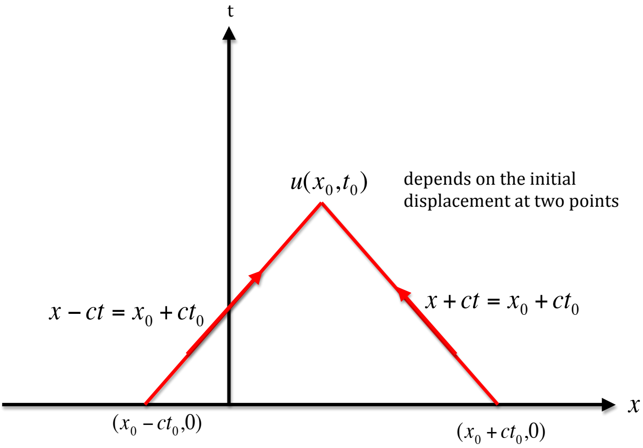
Here, the D'Alembert solution is

$$u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right].$$

- The solution $u$ at a point $(x_0, t_0)$ can be interpreted as being the average of the initial displacement $f(x)$ at the points $(x_0 - ct_0, 0)$ and $(x_0 + ct_0, 0)$ found by backtracking along the lines (characteristic curves)

$$x - ct = x_0 - ct_0$$
$$x + ct = x_0 + ct_0$$

# Fig.12.1 Interpretation of $u(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)]$ in the $xt$-plane



For example, using this interpretation, the IVP

**Problem 12-2**

To find the function $u(x,t)$ that satisfies

PDE: $$u_{tt} = c^2 u_{xx}, \qquad -\infty < x < \infty, \; 0 < t < \infty$$

ICs: $$\begin{cases} u(x,0) = \begin{cases} 1, & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases} & -\infty < x < \infty \\ u_t(x,0) = 0 \end{cases}$$

would give us the solution in the $xt$-plane shown in Fig. 12.2.

# Fig. 12.2 Solution of problem 12-2 in the *xt*-plane