

PDE and Boundary-Value Problems

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Lecture 17

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Purpose of Lesson

- To explain how PDEs that don't involve the time derivative occur in nature. These differential equations have no **initial conditions**, but only boundary conditions.
- To discuss the most common types of BCs for elliptic-type problems.
- To show how to solve the interior Dirichlet problem for the circle by separation of variables.
- To discuss an **alternative** integral-form of a solution (**Poisson integral formula**) to the interior Dirichlet problem.

Chapter 4. Elliptic-Type Problems

- Until now, the problems we've discussed involved phenomena that changed over space **and** time. There are, however, many important problems whose outcomes do not change with time, but only with respect to space.
- These problems, for the most part, are described by **elliptic boundary-value problems**.
- There are two common situations that give rise to PDEs that don't involve time; they are
 - 1 Steady-state problems.
 - 2 Problems where we factor out the time component in the solution.

Steady-State Problems

- Steady-state solution is a solution when $t \rightarrow \infty$.
- It's obvious if the solution **doesn't change in time**, then $u_t = 0$. To find the steady-state solution $u(x, \infty)$ (if it exists), we let $u_t = 0$ and solve the corresponding BVP.
- For some problems, a steady-state solution may not exist.

Factoring out the Time Component in Hyperbolic and Parabolic Problems

- In the circular drumhead problem we looked for solutions of the form $u(r, \theta, t) = U(r, \theta)T(t)$ which yielded the Helmholtz BVP

$$\begin{aligned}\text{PDE: } \Delta U + \lambda^2 U &= 0 \\ \text{BC: } U(1, \theta) &= 0\end{aligned}$$

This situation is common in PDEs where the solution represents a **shape** factor $U(r, \theta)$ multiplied by a time factor $T(t)$.

- As a matter of fact, we arrive at the same Helmholtz equation by factoring out the time component in the **heat equation**.

The Three Main Types of BCs in BVPs

There are three types of BCs that are most common for elliptic-type problems.

- 1. BVPs of the First Kind (Dirichlet Problems)
 - The PDE holds over the given region of space, and the solution is **specified** on the boundary of the region.
 - There are **interior** and **exterior** Dirichlet problems.
 - Dirichlet problems are common in electrostatics when we want to find the potential in a region with the potential given on the boundary.

● 2. BVPs of the Second Kind (Neumann Problems)

- The PDE holds in some region of space, but now the **outward normal derivative** $\frac{\partial u}{\partial n}$ (which is **proportional to the inward flux**) is specified on the boundary.
- Neumann problems are common in steady-state heat flow and electrostatics, where the flux (in heat energy, electrons, and so forth) is given over the boundary.
- Neumann problems make sense only if the net gain across the boundary is zero. Mathematically, this says that

$$\int_C \frac{\partial u}{\partial n} = 0.$$

Otherwise the problem has no solution.

- The Neumann problem differs from other BCs in that solutions are not unique.

● 3. BVPs of the Third Kind (Mixture Problems)

- These problems correspond to the PDEs being given in some region of space, but now the condition on the boundary is a mixture of the first two kinds

$$\frac{\partial u}{\partial n} + h(u - g) = 0,$$

where h is a constant (input to the problem) and g is a given function that can vary over the boundary.

- A more suggestive form of this BC would be

$$\frac{\partial u}{\partial n} = -h(u - g)$$

which says the **inward flux** across the boundary is proportional to the **difference** between the temperature u and some specified temperature g .

The Dirichlet Problems

There are many regions of interest where we might solve the Dirichlet problem. Just to name a few, we could have the Dirichlet problem:

- Inside a circle
- In an annulus
- Outside a circle
- Inside a sphere
- Between two spheres
- Between two lines (in two dimensions)
- Between two planes (in three dimensions)

Remarks

- The list of the Dirichlet problems is endless.
- Our intention is to solve a representative sample of Dirichlet problems and learn the general principles.

Interior Dirichlet Problem for a Circle

Problem 17-1

To find the function $u(r, \theta)$ that satisfies

$$\text{PDE: } u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad 0 < r < 1$$

$$\text{BC: } u(1, \theta) = g(\theta), \quad 0 \leq \theta < 2\pi.$$

Remarks

- Problem 17-1 can be interpreted as finding the electrostatic potential inside a circle when the potential is given on the boundary.
- Another application is the soap film model. If we start with a circular wire hoop and distort it so that the distortion is measured by $g(\theta)$ and dip it into a soap solution, a film of soap is formed within the wire. The height of the film is represented by the solution of problem 17-1, provided the displacement $g(\theta)$ is small.

Step 1.

- Substituting $u(r, \theta) = R(r)\Theta(\theta)$ into the Laplace equation we get

$$r^2 R'' + rR' - \lambda^2 R = 0 \quad (\text{Euler's equation})$$

$$\Theta'' + \lambda^2 \Theta = 0$$

- The separation constant λ must be **nonnegative**. If the separation constant were negative, the function $\Theta(\theta)$ would not be periodic.
- If $\lambda = 0$, then Euler's equation reduces to

$$r^2 R'' + rR' = 0$$

and it is easy to see that the general solution is

$$R(r) = a + b \ln r, \quad \Theta(\theta) = \beta + \gamma \theta.$$

Since we are looking for bounded and periodic solution, we would throw out the terms $\ln r$ and $\gamma \theta$. So, in this case

$$u(r, \theta) = \text{const.}$$

Step 1. (cont.)

- If $\lambda^2 > 0$, then Euler's equation is

$$r^2 R'' + rR' - \lambda^2 R = 0$$

and to solve this, we look for solutions of the form $R(r) = r^\alpha$. The goal is to find two values of α (say α_1 and α_2) so that the general solution will be

$$R(r) = ar^{\alpha_1} + br^{\alpha_2}.$$

Plugging $R(r) = r^\alpha$ into Euler's equation yields $\alpha = \pm\lambda$ and, hence,

$$R(r) = ar^\lambda + br^{-\lambda}.$$

- For $\lambda^2 > 0$ the solution of the equation

$$\Theta''(\theta) + \lambda^2 \Theta = 0$$

has the form

$$\Theta(\theta) = \beta \cos(\lambda\theta) + \gamma \sin(\lambda\theta).$$

Step 1. (cont.)

- Since Θ should be periodic, we conclude that $\lambda = 0, 1, 2, \dots$. Consequently, all solutions $R(r)\Theta(\theta)$ has the form

$$u_n(r, \theta) = r^n [a_n \cos(n\theta) + b_n \sin(n\theta)].$$

Thus,

$$u(r, \theta) = \sum_{n=0}^{\infty} r^n [a_n \cos(n\theta) + b_n \sin(n\theta)]. \quad (17.1)$$

Step 2.

- Substituting (17.1) into the BC, we conclude

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} g(\alpha) d\alpha,$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} g(\alpha) \cos(n\alpha) d\alpha,$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} g(\alpha) \sin(n\alpha) d\alpha.$$

Step 3.

- To summarize, the solution to the interior Dirichlet problem 17-1 is

$$u(r, \theta) = \sum_{n=0}^{\infty} r^n [a_n \cos(n\theta) + b_n \sin(n\theta)]. \quad (17.2)$$

Observations on the Dirichlet Solution:

- The interpretation of our solution (17.2) is that we should expand boundary function $g(\theta)$ as a Fourier series

$$g(\theta) = \sum_{n=0}^{\infty} [a_n \cos(n\theta) + b_n \sin(n\theta)]$$

and then solve the problem for each sine and cosine in the series.

Since each of these terms will give rise to solutions $r^n \sin(n\theta)$ and $r^n \cos(n\theta)$, we can then say (by [superposition](#)) that

$$u(r, \theta) = \sum_{n=0}^{\infty} r^n [a_n \cos(n\theta) + b_n \sin(n\theta)].$$

- The solution of

$$\text{PDE: } \Delta u = 0, \quad 0 < r < 1$$

$$\text{BC: } u(1, \theta) = 1 + \sin \theta + \frac{1}{2} \sin(3\theta) + \cos(4\theta), \quad 0 \leq \theta < 2\pi.$$

would be

$$u(r, \theta) = 1 + r \sin \theta + \frac{r^3}{2} \sin(3\theta) + r^4 \cos(4\theta).$$

Here, the $g(\theta)$ is already in the form of a Fourier series, with

$$a_0 = 1$$

$$b_1 = 1$$

$$a_4 = 1$$

$$b_3 = 0.5$$

$$\text{All other } a_n\text{'s} = 0$$

$$\text{All other } b_n\text{'s} = 0$$

and so we don't have to use the formulas for a_n and b_n .

- If the radius of the circle were arbitrary (say R), then the solution would be

$$u(r, \theta) = \sum_{n=0}^{\infty} \left(\frac{r}{R}\right)^n [a_n \cos(n\theta) + b_n \sin(n\theta)].$$

- Note that the constant term a_0 in solution (17.2) represents the **average of g**

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} g(\alpha) d\alpha.$$

- We start with the separation of variables solution

$$u(r, \theta) = \sum_{n=0}^{\infty} \left(\frac{r}{R}\right)^n [a_n \cos(n\theta) + b_n \sin(n\theta)]$$

(we now take an arbitrary radius for the circle) and substitute the coefficients a_n and b_n .

- After a few manipulations involving algebra, calculus, and trigonometry, we have

$$\begin{aligned}
u(r, \theta) &= \frac{1}{2\pi} \int_0^{2\pi} g(\theta) d\theta \\
&+ \frac{1}{\pi} \sum_{n=1}^{\infty} \left(\frac{r}{R}\right)^n \int_0^{2\pi} g(\alpha) [\cos(n\alpha) \cos(n\theta) + \sin(n\alpha) \sin(n\theta)] d\alpha \\
&= \frac{1}{2\pi} \int_0^{2\pi} \left\{ 1 + 2 \sum_{n=1}^{\infty} \left(\frac{r}{R}\right)^n \cos[n(\theta - \alpha)] \right\} g(\alpha) d\alpha \\
&= \frac{1}{2\pi} \int_0^{2\pi} \left\{ 1 + \sum_{n=1}^{\infty} \left(\frac{r}{R}\right)^n [e^{in(\theta-\alpha)} + e^{-in(\theta-\alpha)}] \right\} g(\alpha) d\alpha \\
&= \frac{1}{2\pi} \int_0^{2\pi} \left\{ 1 + \frac{re^{i(\theta-\alpha)}}{R - re^{i(\theta-\alpha)}} + \frac{re^{-i(\theta-\alpha)}}{R - re^{-i(\theta-\alpha)}} \right\} g(\alpha) d\alpha
\end{aligned}$$

$$\begin{aligned} u(r, \theta) &= \frac{1}{2\pi} \int_0^{2\pi} \left\{ 1 + \frac{re^{i(\theta-\alpha)}}{R - re^{i(\theta-\alpha)}} + \frac{re^{-i(\theta-\alpha)}}{R - re^{-i(\theta-\alpha)}} \right\} g(\alpha) d\alpha \\ &= \frac{1}{2\pi} \int_0^{2\pi} \left[\frac{R^2 - r^2}{R^2 - 2rR \cos(\theta - \alpha) + r^2} \right] g(\alpha) d\alpha \end{aligned} \tag{17.3}$$

It is the **Poisson Integral Formula**. So what we have is an alternative form for the solution to the interior Dirichlet problem.

Remarks

- We can interpret the Poisson integral solution (17.3) as finding the potential u at (r, θ) as a **weighted average** of the boundary potentials $g(\theta)$ weighted by the Poisson kernel

$$\text{Poisson kernel} = \frac{R^2 - r^2}{R^2 - 2rR \cos(\theta - \alpha) + r^2}.$$

- For boundary values $g(\alpha)$ close to (r, θ) , the Poisson kernel gets **large**, since the denominator of the Poisson kernel is the square of the distance from (r, θ) to (R, α) .

Remarks (cont.)

- Unfortunately, if (r, θ) is extremely close to the boundary $r = R$, then the Poisson kernel gets very large for those values of α that are closest to (r, θ) . For this reason, when (r, θ) is close to the boundary, the **series** solution works better for evaluating the numerical value of the solution.
- If we evaluate the potential at the center of the circle by the Poisson integral, we find

$$u(0, 0) = \frac{1}{2\pi} \int_0^{2\pi} g(\alpha) d\alpha.$$

In other words, the potential at the center of the circle is the average of the boundary potentials.

Remarks (cont.)

- We can always solve the BVP (nonhomogeneous PDE)

$$\text{PDE: } \Delta u = f, \quad \text{Inside } D$$

$$\text{BC: } u = 0, \quad \text{On the boundary of } D$$

by

- 1 Finding any solution V of $\Delta V = f$ (A particular solution).
- 2 Solving the new BVP

$$\text{PDE: } \Delta W = 0, \quad \text{Inside } D$$

$$\text{BC: } W = V, \quad \text{On the boundary of } D$$

- 3 Observing that $u = V - W$ is our desired solution.

In other words, we can transfer the nonhomogeneity from the PDE to BC.

Remarks (cont.)

- We can solve the BVP (nonhomogeneous BC)

$$\text{PDE: } \Delta u = 0, \quad \text{Inside } D$$

$$\text{BC: } u = f, \quad \text{On the boundary of } D$$

by

- 1 Finding any solution V that satisfies $V = f$ on the boundary of D .
- 2 Solving the new BVP

$$\text{PDE: } \Delta W = \Delta V, \quad \text{Inside } D$$

$$\text{BC: } W = 0, \quad \text{On the boundary of } D$$

- 3 Observing that $u = V - W$ is the solution to our problem.

In other words, we can transfer the nonhomogeneity from the BC to the PDE.