

PDE and Boundary-Value Problems

Winter Term 2016/2017

Lecture 19

Saarland University

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Purpose of Lesson

- To derive the **fundamental solution** of the Laplace equation and discuss how to solve with its help the Poisson equation (nonhomogeneous Laplace equation).
- To show how a nonhomogeneous Dirichlet problem can be solved by the **Green's function** approach (the impulse-response function).
- To derive Green's functions for a half-space and for a ball.
- To show how a PDE can be changed to a system of algebraic equations by replacing the **partial derivatives** in the differential equation with their **finite-difference approximations**. The system of algebraic equations can then be solved numerically by an iterative process in order to obtain an approximate solution to the PDE.

Fundamental Solution

Problem 19-1

To find a function $u(x)$ that satisfies

$$\Delta u = 0 \quad x \in \mathbb{R}^n$$

- We attempt to find a solution u of Laplace's equation in \mathbb{R}^n , having the form

$$u(x) = v(r),$$

where $r = |x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$.

- It is clear that $\Delta u = 0$ if and only if that

$$v''(r) + \frac{n-1}{r}v'(r) = 0.$$

- If $r > 0$ we have

$$v(r) = \begin{cases} a \ln r + b & (n = 2) \\ \frac{a}{r^{n-2}} + b & (n \geq 3), \end{cases}$$

where a and b are constants.

The above consideration motivate the following

Definition

The function

$$\Phi(x) := \begin{cases} -\frac{1}{2\pi} \ln |x|, & (n = 2) \\ \frac{1}{n(n-2)\omega(n)} \frac{1}{|x|^{n-2}}, & (n \geq 3), \end{cases}$$

defined for $x \in \mathbb{R}^n$, $x \neq 0$, is the **fundamental solution** of Laplace's equation.

- $\omega(n)$ denotes the volume of the unit ball in \mathbb{R}^n .
- $|D\Phi(x)| \leq \frac{C}{|x|^{n-1}}$, $|D^2\Phi(x)| \leq \frac{C}{|x|^n}$, ($x \neq 0$)

Theorem (Solving Poisson's equation)

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be twice continuous differentiable with compact support, and let u satisfy

$$\begin{aligned}
 u(x) &= \int_{\mathbb{R}^n} \Phi(x-y)f(y)dy \\
 &= \begin{cases} -\frac{1}{2\pi} \int_{\mathbb{R}^2} \ln|x-y|f(y)dy & (n=2) \\ \frac{1}{n(n-2)\omega(n)} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-2}} dy & (n \geq 3) \end{cases}
 \end{aligned}$$

Then $u \in C^2(\mathbb{R}^n)$ and $-\Delta u = f$ in \mathbb{R}^n .

Remarks

- We **cannot** just compute

$$\Delta u(x) = \int_{\mathbb{R}^n} \Delta_x \Phi(x - y) f(y) dy = 0.$$

- Indeed, $D^2\Phi(x - y)$ is **not** summable near the singularity at $y = x$, and so the differentiation under the integral sign is unjustified (and incorrect).
- We must proceed more carefully in calculating Δu .

A Nonhomogeneous Dirichlet Problem (Green's Function)

Problem 19-2

To find a function $u(x)$ that satisfies

$$\text{PDE: } -\Delta u = f, \quad x \in U \subset \mathbb{R}^n, \quad U - \text{open, bounded, } \partial U \in C^1$$

$$\text{BC: } u = g, \quad x \in \partial U$$

We propose to obtain a general representation formula for the solution of Problem 19-2.

Derivation of Green's function

- Fix $x \in U$, choose $\varepsilon > 0$ so small that $B_\varepsilon(x) \subset U$.
- Applying Green's formula

$$\int_{\Omega} [u\Delta v - v\Delta u] dx = \int_{\partial\Omega} \left[u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right] dS$$

on the region $U_\varepsilon := U - B_\varepsilon(x)$ to $u(y)$ and $\Phi(y - x)$ we get

$$\begin{aligned} & \int_{U_\varepsilon} [u(y)\Delta\Phi(y-x) - \Phi(y-x)\Delta u(y)] dy \\ &= \int_{\partial U_\varepsilon} \left[u(y) \frac{\partial\Phi}{\partial n}(y-x) - \Phi(y-x) \frac{\partial u}{\partial n} \right] dS(y) \end{aligned}$$

Derivation of Green's function (cont.)

• We observe that

① $\Delta\Phi(y - x) = 0$ for $y \neq x$;

②

$$\left| \int_{\partial B_\varepsilon(x)} \Phi(y - x) \frac{\partial u}{\partial n}(y) dS(y) \right| \leq C\varepsilon^{n-1} \max_{\partial B_\varepsilon(x)} |\Phi| = o(1)$$

as $\varepsilon \rightarrow 0$;

③

$$\int_{\partial B_\varepsilon(x)} u(y) \frac{\partial \Phi}{\partial n}(y - x) dS(y) = \int_{\partial B_\varepsilon(x)} u(y) dS(y) \rightarrow u(x)$$

as $\varepsilon \rightarrow 0$;

Derivation of Green's function (cont.)

- Hence our sending $\varepsilon \rightarrow 0$ yields the formula:

$$u(x) = \int_{\partial U} \left[\Phi(y-x) \frac{\partial u}{\partial n}(y) - u(y) \frac{\partial \Phi}{\partial n}(y-x) \right] dS(y) - \int_U \Phi(y-x) \Delta u(y) dy$$

The above identity is valid for any point $x \in U$ and any function $u \in C^2(\bar{U})$

Derivation of Green's function (cont.)

$$\begin{aligned}
 u(x) = \int_{\partial U} \left[\Phi(y-x) \frac{\partial u}{\partial n}(y) - u(y) \frac{\partial \Phi}{\partial n}(y-x) \right] dS(y) \\
 - \int_U \Phi(y-x) \Delta u(y) dy
 \end{aligned} \tag{19.1}$$

Remarks

- If we apply formula (19.1) to problem 19-2, we see that the normal derivative $\frac{\partial u}{\partial n}$ along ∂U is unknown to us.
- We must somehow modify (19.1) to remove this term.

Derivation of Green's function (cont.)

The idea is now to introduce for fixed x a **corrector** function $\phi^x = \phi^x(y)$, solving the BVP:

Problem 19-3

To find a function $\phi^x(y)$ that satisfies

$$\text{PDE: } \Delta \phi^x = 0, \quad y \in U$$

$$\text{BC: } \phi^x = \Phi(y - x), \quad y \in \partial U$$

Derivation of Green's function (cont.)

Applying Green's formula once more, we compute

$$\begin{aligned}
 - \int_U \phi^x(y) \Delta u(y) dy &= \int_{\partial U} \left[u(y) \frac{\partial \phi^x}{\partial n}(y) - \phi^x(y) \frac{\partial u}{\partial n}(y) \right] dS(y) \\
 &= \int_{\partial U} \left[u(y) \frac{\partial \phi^x}{\partial n}(y) - \Phi(y-x) \frac{\partial u}{\partial n}(y) \right] dS(y)
 \end{aligned}$$

We introduce next this

Definition

Green's function for the region U is

$$G(x, y) := \Phi(y-x) - \phi^x(y) \quad (x, y \in U, \quad x \neq y).$$

Adopting this terminology, we find from (19.1) and the above identity for ϕ^x the formula

$$u(x) = - \int_{\partial U} u(y) \frac{\partial G}{\partial n}(x, y) dS(y) - \int_U G(x, y) \Delta u(y) dy, \quad (x \in U) \quad (19.2)$$

where

$$\frac{\partial G}{\partial n}(x, y) = n(y) \cdot D_y G(x, y)$$

is the outer normal derivative of G with respect to the variable y .

Theorem (Representation formula using Green's function)

If $u \in C^2(\bar{U})$ solves problem 19-2, then

$$u(x) = - \int_{\partial U} g(y) \frac{\partial G}{\partial n}(x, y) dS(y) + \int_U f(y) G(x, y) dy \quad (x \in U)$$

- To construct Green's function G for the given domain U is in general a difficult matter, and can be done only when U has simple geometry
- We will build Green's functions for two regions with simple geometry, namely the half-space \mathbb{R}_+^n and the unit ball $B_1(0)$.

Theorem (Representation formula using Green's function)

If $u \in C^2(\bar{U})$ solves the problem

$$\text{PDE: } -\Delta u = f, \quad x \in U \subset \mathbb{R}^n, \quad U - \text{open, bounded, } \partial U \in C^1$$

$$\text{BC: } u = g, \quad x \in \partial U$$

then

$$u(x) = - \int_{\partial U} g(y) \frac{\partial G}{\partial n}(x, y) dS(y) + \int_U f(y) G(x, y) dy \quad (x \in U),$$

where

$$G(x, y) := \Phi(y - x) - \phi^x(y) \quad (x, y \in U, \quad x \neq y).$$

Green's Function for a Half-Space

- We set for $x, y \in \mathbb{R}_+^n$

$$\phi^x(y) := \Phi(y - \tilde{x}) = \Phi(y_1 - x_1, \dots, y_{n-1} - x_{n-1}, y_n + x_n).$$

The idea is that the corrector ϕ^x is built from Φ by „reflecting the singularity“ from $x \in \mathbb{R}_+^n$ to $\tilde{x} \notin \mathbb{R}_+^n$.

- If $y \in \partial\mathbb{R}_+^n$ then

$$\phi^x(y) = \Phi(y - x),$$

and thus

$$\begin{cases} \Delta\phi^x = 0 & \text{in } \mathbb{R}_+^n \\ \phi^x = \Phi(y - x) & \text{on } \partial\mathbb{R}_+^n \end{cases}$$

as required.

Green's Function for a Half-Space (cont.)

- Therefore, Green's function for the half-space \mathbb{R}_+^n is

$$G(x, y) := \Phi(y - x) - \Phi(y - \tilde{x}), \quad (x, y \in \mathbb{R}_+^n, x \neq y)$$

- It is evident that

$$\begin{aligned} \frac{\partial G}{\partial y_n}(x, y) &= \frac{\partial \Phi}{\partial y_n}(y - x) - \frac{\partial \Phi}{\partial y_n}(y - \tilde{x}) \\ &= \frac{-1}{n\omega(n)} \left[\frac{y_n - x_n}{|y - x|^n} - \frac{y_n + x_n}{|y - \tilde{x}|^n} \right] \end{aligned}$$

- So, if $y \in \partial\mathbb{R}_+^n$

$$\frac{\partial G}{\partial n}(x, y) = -\frac{\partial G}{\partial y_n}(x, y) = -\frac{-2x_n}{n\omega(n)} \frac{1}{|x - y|^n}.$$

Green's function for a half-space (cont.)

Suppose now u solves the BVP

Problem 19-4

To find a function $u(x)$ that satisfies

$$\text{PDE: } \Delta u = 0, \quad x \in \mathbb{R}_+^n$$

$$\text{BC: } u = g, \quad x \in \partial\mathbb{R}_+^n$$

We expect

$$u(x) = \frac{2x_n}{n\omega(n)} \int_{\partial\mathbb{R}_+^n} \frac{g(y)}{|x-y|^n} dy \quad (x \in \mathbb{R}_+^n) \quad (19.3)$$

to be a representation formula for our solution.

Green's Function for a Half-Space (cont.)

Remark

We must check directly that formula (19.3) provides us with a solution of the BVP 19-4, i.e., we must check that

$$u \in C^2(\overline{\mathbb{R}_+^n}),$$

and

$$\Delta u = 0 \quad \text{in } \mathbb{R}_+^n,$$

and

$$\lim_{\mathbb{R}_+^n \ni x \rightarrow x^0} u(x) = g(x^0) \quad \text{for each point } x^0 \in \partial \mathbb{R}_+^n.$$

Green's Function for a Ball

We solve problem for the unit ball, i.e.,

$$U = B_1(0) = \{x \in \mathbb{R}^n : |x| < 1\}$$

- We set for $x, y \in B_1(0)$

$$\phi^x(y) := \Phi(|x|(y - \tilde{x})), \quad \tilde{x} = \frac{x}{|x|^2}$$

Again, the idea is to „invert the singularity“ from $x \in B_1(0)$ to $\tilde{x} \notin B_1(0)$.

Green's Function for a Ball (cont.)

- Assume for the moment $n \geq 3$.
 - The mapping $y \mapsto \Phi(y - \tilde{x})$ is harmonic for $y \neq \tilde{x}$
 - Thus $y \mapsto |x|^{2-n}\Phi(y - \tilde{x})$ is also harmonic for $y \neq \tilde{x}$

Therefore, $\phi^x(y) := \Phi(|x|(y - \tilde{x}))$ is harmonic in $B_1(0)$.

- If $y \in \partial B_1(0)$ and $x \neq 0$, then

$$\begin{aligned} |x|^2|y - \tilde{x}|^2 &= |x|^2 \left(|y|^2 - \frac{2y \cdot x}{|x|^2} + \frac{1}{|x|^2} \right) \\ &= |x|^2 - 2y \cdot x + 1 = |x - y|^2. \end{aligned}$$

Thus, $(|x||y - \tilde{x}|)^{-(n-2)} = |x - y|^{-(n-2)}$. Consequently

$$\phi^x(y) = \Phi(y - x) \quad (y \in \partial B_1(0))$$

as required.

Green's Function for a Ball (cont.)

- Therefore, Green's function for the unit ball $B_1(0)$ is

$$G(x, y) := \Phi(y - x) - \Phi(|x|(y - \tilde{x})) \quad (x, y \in B_1(0), x \neq y)$$

- If $y \in \partial B_1(0)$ then

$$\begin{aligned} \frac{\partial G}{\partial n}(x, y) &= \sum_{i=1}^n y_i \frac{\partial G}{\partial y_i}(x, y) \\ &= \frac{-1}{n\omega(n)} \frac{1}{|x - y|^n} \sum_{i=1}^n y_i \left((y_i - x_i) - y_i |x|^2 + x_i \right) \\ &= \frac{-1}{n\omega(n)} \frac{1 - |x|^2}{|x - y|^n}. \end{aligned}$$

Green's Function for a Ball (cont.)

Suppose now u solves the BVP

Problem 19-5

To find a function $u(x)$ that satisfies

$$\text{PDE: } \Delta u = 0, \quad x \in B_1(0)$$

$$\text{BC: } u = g, \quad x \in \partial B_1(0)$$

We expect

$$u(x) = \frac{1 - |x|^2}{n\omega(n)} \int_{\partial B_1(0)} \frac{g(y)}{|x - y|^n} dS(y) \quad (x \in B_1(0)) \quad (19.4)$$

to be a representation formula for our solution.