PDE and Boundary-Value Problems Winter Term 2016/2017

Lecture 2

Universität des Saarlandes

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Purpose of Lesson

- To continue the dicussion about classification of 2nd order PDEs and define the normal (canonical) forms of 2nd order PDEs in two variables.
- To introduce the notions of BVPs and classical solutions.

Normal forms of 2nd order PDEs in two independent variables:

Using a suitable transformation of independent variables

$$\xi = \xi(x, y), \qquad \eta = \eta(x, y)$$

we can always reduce equation

$$a(x, y)u_{xx} + b(x, y)u_{xy} + c(x, y)u_{yy} + f(x, y, u, u_x, u_y) = 0$$

to one of the following three NORMAL FORMs:

- for hyperbolic equations

$$u_{\xi\eta} = F(\xi, \eta, u, u_\xi, u_\eta), \quad \text{or} \quad u_{\xi\xi} - u_{\eta\eta} = F(\xi, \eta, u, u_\xi, u_\eta);$$

- for parabolic equations

$$u_{\eta\eta} = F(\xi, \eta, u, u_\xi, u_\eta),$$

  where F **must** depend on u_ξ: otherwise the equation degenerates into an ODE;

- for elliptic equations

$$u_{\xi\xi} + u_{\eta\eta} = F(\xi, \eta, u, u_\xi, u_\eta).$$

The classification (elliptic, parabolic etc.) can be extended to equations depending on more than 2 variables.

Consider the 2nd order PDE depeding on n variables,

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} u_{x_i x_j} + \sum_{i=1}^{n} b_i u_{x_i} + cu + g = 0.$$

The coefficient matrix (a_{ij}) should be symmetrized because

$$\frac{\partial^2}{\partial x_i \partial x_j} = \frac{\partial^2}{\partial x_j \partial x_i}, \qquad \text{for any} \quad i \text{ and } j \text{ in } [1, n].$$

The classification is as follows:

- hyperbolic for ($Z = 0$ and $P = 1$) or ($Z = 0$ and $P = n - 1$)
- parabolic for $Z > 0$ ($\Leftrightarrow \det(a_{ij}) = 0$)
- elliptic for ($Z = 0$ and $P = n$) or ($Z = 0$ and $P = 0$)
- ultra-hyperbolic for ($Z = 0$ and $1 < P < n - 1$)

where

$$\begin{aligned} Z &= \text{ number of zero eigenvalues } (a_{ij}), \\ P &= \text{ number of strictly positive eigenvalues of } (a_{ij}). \end{aligned}$$

What are Boundary Value and Initial Value Problems?

PDEs can have many very different solutions. For example, the Laplace equation

$$u_{xx} + u_{yy} = 0$$

is solved by

$$u = x^2 - y^2, \quad u = e^x \cos y, \quad u = \ln\left(x^2 + y^2\right).$$

General solutions of higher-order PDEs are often difficult to find and hard to use.

A unique solution modelling a given processes or phenomenon can be specified by additional constraints imposed on the boundary of the region in space (boundary condition) or at some time (initial conditions).

Remarks

- If the number of such constraints is too large, the problem will be overdetermined and will not have any solutions.
- If the number of constraints is too small, the problem will have more than one solution.

If a differntial equation has been given together with all necessary boundary and/or initial conditions, it is said that a boudary value or initial value problem has been formulated.

Common IVPs, formulated in a region $\mathbb{R}^n \times [0; +\infty)$ are as follows:

- The Cauchy problem: Determine a function u such that

$$\begin{cases} \text{PDE} \quad \text{in } \mathbb{R}^n \times [0, +\infty) \\ u\big|_{t=0} = \varphi(x) \end{cases} \quad \text{or} \quad \begin{cases} \text{PDE} \quad \text{in } \mathbb{R}^n \times [0, +\infty) \\ u\big|_{t=0} = \varphi(x) \\ u_t\big|_{t=0} = \psi(x), \end{cases}$$

where φ and ψ are given functions, defined in $\mathbb{R}^n$.

Common BVPs, formulated in a region Ω in space, (Ω has the boundary $\partial\Omega$) are as follows:

- The Dirichlet problem: Determine a function u such that

$$\begin{cases} \text{PDE} & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

  where φ is a given function defined on $\partial\Omega$.

- The Neumann problem: Determine a function u such that

$$\begin{cases} \text{PDE} & \text{in } \Omega \\ \dfrac{\partial u}{\partial n} = \psi & \text{on } \partial\Omega, \end{cases}$$

  where ψ is a given function defined on $\partial\Omega$ and $\frac{\partial u}{\partial n}$ is the normal derivative.

We say that a given BVP is well-posed if

- the BVP in fact has a solution;
- this solution is unique;
- the solution depends continuosly on the data given in the problem.

Remark

The last condition is particulary important for problems arising from physical applications: we would prefer that our (unique) solution changes only a little when the conditions specifying the problem change a little.

How regular should be a solution?

Should we ask, for example, that a solution u of our BVP must be real analytic or at least infinitely differentiable?

This might be desirable, but perhaps we are asking too much.

It would be more practical to require a solution of the BVP with PDE of order k to be at least k times continuously differentiable.

Then at least all the derivatives which appear in the statement of the PDE will exist and be continuous, although maybe certain higher derivatives will not exist.

We call a solution with this much smoothness a classical solution.