PDE and Boundary-Value Problems
Winter Term 2016/2017

Lecture 5

Saarland University

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Purpose of Lesson

- To introduce the powerful method of separation of variables and to show how this method can be used to solve some of diffusioin-type problems

Separation of Variables

Separation of variables is one of the oldest techniques for solving IBVPs. It applies to problems where

1. PDE is linear and homogeneous (not necessarily constant coefficients).
2. BCs are also linear and homogeneous.

Example of admissible BCs

$$\alpha u_x(0,t) + \beta u(0,t) = 0$$
$$\gamma u_x(L,t) + \delta u(L,t) = 0$$

Method of separation of variables dates back to the time of Joseph Fourier (in fact, it's occasionally called *Fourier's method*).

- Jean Baptiste Joseph Fourier (1768-1830)

Field:	Mathematician, physisist and historian
Alma mater:	École Normale
Advisor:	Joseph Lagrange
Doctoral students:	Gustav Dirichlet, Claude-Louis Navier
Known for:	Fourier series, Fourier transform, Fourier's law of conduction

Separation of Variables

It is probably the most widely used method of solution (when applicable).

Overview of Separation of Variables

Separation of variables looks for simple-type solutions to the PDE of the form

$$u(x,t) = X(x)T(t)$$

where $X(x)$ is some function of x and $T(t)$ is some of t.

The solutions are simple because any temperature $u(x,t)$ of this form will retain its basic „shape“ for different values of time t.

Overview of Separation of Variables $u(x,t) = X(x)T(t)$

- The general idea is that it is possible to find an infinite number of these solutions to the PDE (which, at the same time, also satisfy the BCs).
- These simple functions

$$u_n(x,t) = X_n(x)T_n(t)$$

(called fundamental solutions) are the building blocks of our problem.

- We are looking for the solution $u(x,t)$ of our IBVP as the resulting sum

$$u(x,t) = \sum_{n=1}^{\infty} A_n X_n(x) T_n(t)$$

which satisfies the initial condition.

Problem 5-1

To find the function $u(x,t)$ that satisfies

$$\text{PDE:} \quad u_t = \alpha^2 u_{xx}, \qquad 0 < x < 1, \quad 0 < t < \infty$$

$$\text{BCs:} \quad \begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases}, \qquad 0 < t < \infty$$

$$\text{IC:} \quad u(x,0) = \phi(x), \qquad 0 \leqslant x \leqslant 1$$

Step 1 (Finding elementary solutions to the PDE)

- We look for solutions of the form $u(x,t) = X(x)T(t)$ by substituting $X(x)T(t)$ into the PDE and solving for $X(x)T(t)$. As a result we get

$$\boxed{X(x)T'(t) = \alpha^2 X''(x)T(t).} \tag{5.1}$$

- If we divide each side of (5.1) by $\alpha^2 X(x)T(t)$, we have

$$\frac{T'(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)} \tag{5.2}$$

and obtain what is called separated variables, that is, the left-hand side of (5.2) depends only on t and the right-hand side of (5.2), only on x.

Step 1 (Finding elementary solutions to the PDE)

- Since x and t are independent of each other, each side must be a fixed constant (say k); hence, we can write

$$T'(t) - k\alpha^2 T(t) = 0$$
$$X''(x) - kX(x) = 0.$$

- Now we can solve each of these two ODEs, multiply them together to get a solution to the PDE (note that we have changed a second-order PDE to two ODEs).

Step 1 (Finding elementary solutions to the PDE)

- We change $k := -\lambda^2$. Otherwise, $T(t)$ factor doesn't go to zero as $t \to \infty$. As a result we get

$$\begin{aligned} T'(t) + \lambda^2\alpha^2 T(t) = 0 \\ X''(x) + \lambda^2 X(x) = 0. \end{aligned} \tag{5.3}$$

- Both of equations in (5.3) are standard-type ODEs and have solutions

$$\begin{aligned} T(t) &= C_1 e^{-\lambda^2\alpha^2 t} \quad (C_1 \text{ an arbitrary constant}) \\ X(x) &= C_2 \sin(\lambda x) + C_3 \cos(\lambda x) \quad (C_2, C_3 \text{ arbitrary}). \end{aligned}$$

Step 1 (Finding elementary solutions to the PDE)

- Hence all functions

$$u(x,t) = e^{-\lambda^2\alpha^2 t}\left(A\sin(\lambda x) + B\cos(\lambda x)\right)$$

(with A, B and λ arbitrary) satisfies the PDE $u_t = \alpha^2 u_{xx}$.

Step 2 (Finding solutions to the PDE and the BCs)

- The next step is to choose a certain subset of solutions

$$e^{-\lambda^2\alpha^2 t}\left(A\sin\left(\lambda x\right) + B\cos\left(\lambda x\right)\right) \tag{5.4}$$

that satisfy the boundary conditions

$$u(0,t) = 0, \qquad u(1,t) = 0 \qquad \forall t > 0.$$

- To do this, we substitute solutions (4.4) into BCs, getting

$$u(0,t) = Be^{-\lambda^2\alpha^2 t} = 0 \quad \Rightarrow \quad B = 0$$
$$u(1,t) = Ae^{-\lambda^2\alpha^2 t}\sin\lambda = 0 \Rightarrow \sin\lambda = 0 \Rightarrow \lambda_n = \pm n\pi.$$

- Case $A = B = 0$ is not interesting.

Step 2 (Finding solutions to the PDE and the BCs)

- We have now finished the second step; we have found an infinite number of functions

$$\boxed{u_n(x,t) = A_n e^{-(n\pi\alpha)^2 t} \sin(n\pi x) \qquad n = 1, 2, 3, \dots}$$

each one satisfying the PDE and BCs.

- These functions u_n are the building blocks (fundamental solutions) of the problem, and our desired solution will be a certain sum of these simple functions; the specific sum will depend on the initial conditions.

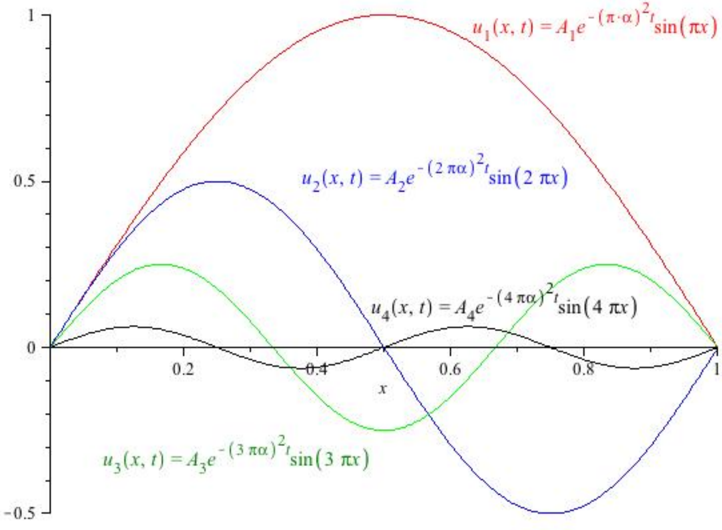

Figure 1: Fundamental solutions

Step 3 (Finding solutions to the PDE, BCs, and the IC)

- The last step is to add the fundamental solutions

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-(n\pi\alpha)^2 t} \sin(n\pi x) \tag{5.5}$$

  in such a way (pick the coefficients A_n) that the initial condition $u(x,0) = \phi(x)$ is satisfied.

- Substituting (5.5) into the IC gives

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin(n\pi x).$$

Step 3 (Finding solutions to the PDE, BCs, and the IC)

Question:

Is it possible to expand the initial temperature $\phi(x)$ as the sum of the elementary function as follows:

$$A_1 \sin(\pi x) + A_2 \sin(2\pi x) + A_3 \sin(3\pi x) + \ldots ?$$

Answer:

The answer to this question is YES provided $\phi(x)$ is continuous.

Hence, the question now becomes how to find the coefficients A_n.

Remark

The functions

$$\sin(n\pi x), \quad n = 1, 2, \ldots$$

are orthogonal to each other in the sense

$$\int_0^1 \sin(m\pi x)\sin(n\pi x)dx = \begin{cases} 0, & m \neq n \\ 1/2, & m = n \end{cases}$$

This property can be illustrated by looking at the graph of these functions.

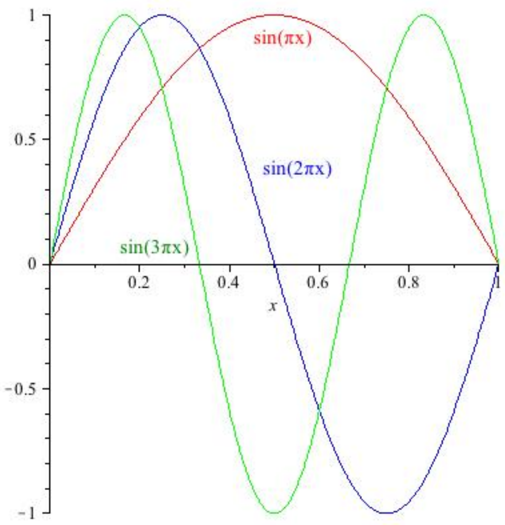

Figure 2: Orthogonal sequence of functions

Step 3 (Finding solutions to the PDE, BCs, and the IC)

We are now in position to solve for the coefficients in the expression

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin(n\pi x). \tag{5.6}$$

- We multiply each side of (5.6) by $\sin(m\pi x)$ (m is an arbitrary integer) and integrate from zero to one. As a result we get

$$\int_0^1 \phi(x) \sin(m\pi x) dx = A_m \int_0^1 sin^2(m\pi x) dx = \frac{1}{2} A_m$$

$$\Rightarrow \qquad A_m = 2 \int_0^1 \phi(x) \sin(m\pi x) dx.$$

Step 3 (Finding solutions to the PDE, BCs, and the IC)

We are done; the solution is

$$\boxed{u(x,t) = 2\sum_{n=1}^{\infty}\left[\int_0^1 \phi(s)\sin(n\pi s)ds\right] e^{-(n\pi\alpha)^2 t}\sin(n\pi x)}.$$

Remark

The terms in the solution

$$u(x,t) = A_1 e^{-(\pi\alpha)^2 t} \sin(\pi x) + A_2 e^{-(2\pi\alpha)^2 t} \sin(2\pi x) + \dots$$

become small very fast due to the factor

$$e^{-(n\pi\alpha)^2 t}.$$

Hence, for long time periods, the solution is approximately equal to the first term

$$u(x,t) \cong A_1 e^{-(\pi\alpha)^2 t} \sin(\pi x).$$