# PDE and Boundary-Value Problems
# Winter Term 2015/2016

Lecture 7

Saarland University

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## Purpose of Lesson

- To show how to solve the IBVP with nonhomogeneous PDE by the eigenfunction expansion method.
- To introduce the idea of integral transforms and show how they transform PDEs in $n$ variables into differential equations in $n-1$ variables.
- To introduce the sine and cosine transforms and use them to solve an infinite-diffusion problem.

# Solving Nonhomogeneous PDEs (Eigenfunction Expansions)

We have discussed how transform nonhomogeneous BCs into homogeneous ones. Unfortunately, the PDE was left nonhomogeneous by this process and we were left with the problem

**Problem 7-1**

$$\text{PDE:} \qquad u_t = \alpha^2 u_{xx} + f(x,t), \qquad 0 < x < 1, \quad 0 < t < \infty$$

$$\text{BCs:} \quad \begin{cases} \alpha_1 u_x(0,t) + \beta_1 u(0,t) = 0 \\ \alpha_2 u_x(1,t) + \beta_2 u(1,t) = 0 \end{cases}, \qquad 0 < t < \infty$$

$$\text{IC:} \qquad u(x,0) = \phi(x), \qquad 0 \leqslant x \leqslant 1$$

# Solving Nonhomogeneous PDEs (Eigenfunction Expansions)

We solve problem 7-1 by a method that is analogous to the method of *variation of parameters* in ODEs and is known as the eigenfunction expansion method.

The idea is quite simple. The solution of problem 7-1 with $f(x,t) = 0$ (corresponding homogeneous problem) is given by

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-(\lambda_n \alpha)^2 t} X_n(x),$$

where $\lambda_n$ and $X_n(x)$ are the eigenvalues and eigenfunctions of the Sturm-Liouville problem,

$$\begin{aligned} X'' + \lambda^2 X &= 0 \\ \alpha_1 X'(0) + \beta_1 X(0) &= 0 \\ \alpha_2 X'(1) + \beta_1 X(1) &= 0 \end{aligned}$$

# Solving Nonhomogeneous PDEs (Eigenfunction Expansions)

We ask whether the solution of the nonhomogeneous problem 7-1 can be written in the slightly more general form

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) X_n(x)?$$

To show how this method works, we apply it to a problem more simple as problem 7-1. So, the details aren't as complicated.

# Solution by the Eigenfunction Expansion Method

Consider the nonhomogeneous problem

**Problem 7-2**

PDE: $u_t = \alpha^2 u_{xx} + f(x,t), \quad 0 < x < 1, \quad 0 < t < \infty$

BCs: $\begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases}, \qquad 0 < t < \infty$

IC: $u(x,0) = \phi(x), \qquad 0 \leqslant x \leqslant 1$

To solve problem 7-2 we divide the procedure into the following steps:

# Step 1 (Find $X_n(x)$, that is, the solutions of the associated SLP)

- We find the functions $X_n(x)$ which are the eigenvectors of the associated Sturm-Liouville system

$$\begin{aligned} X'' + \lambda^2 X &= 0 \\ X(0) &= 0 \\ X(1) &= 0. \end{aligned}$$

It is clear that

$$X_n(x) = \sin(n\pi x), \qquad n = 1, 2, \ldots$$

# Step 2 (Decomposion of $f(x,t)$)

- We decompose the heat source $f(x,t)$ into simple components

$$f(x,t) = f_1(t)X_1(x) + f_2(t)X_2(x) + \dots + f_n(t)X_n(x) + \dots$$

- For problem 7-2, our decomposion of the heat source has the form

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t) \sin(n\pi x). \tag{7.1}$$

# Step 2 (Decomposion of $f(x, t)$)

- To find the functions $f_n(t)$ we merely multiply each side of (7.1) by $\sin(m\pi x)$ and integrate from zero to one (with respect to $x$); hence, we have

$$\int_0^1 f(x,t)\sin(m\pi x)dx = \sum_{n=1}^{\infty} f_n(t)\int_0^1 \sin(m\pi x)\sin(n\pi x)dx$$
$$= \frac{1}{2} f_m(t).$$

- Changing $m$ to $n$ we get

$$f_n(t) = 2\int_0^1 f(x,t)\sin(n\pi x)dx.$$

# Step 3 (Find the responce $u_n(x,t) = T_n(t)X_n(x)$)

- We try to find our solution as a sum of the individual responses

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin(n\pi x);$$

in other words, we seek the functions $T_n(t)$.

# Step 3 (Find the responce $u_n(x,t) = T_n(t)X_n(x)$)

- Substituting $u(x,t) = \sum_{n=1}^{\infty} T_n(t)\sin(n\pi x)$ into the system 7-2 gives us

$$\text{PDE:} \quad \sum_{n=1}^{\infty} \left[ T_n'(t) + (n\pi\alpha)^2 T_n(t) - f_n(t) \right] \sin(n\pi x) = 0$$

$$\text{BCs:} \quad \begin{cases} \sum_{n=1}^{\infty} T_n'(t)\sin(0) = 0 & \text{(says nothing; zero=zero)} \\ \sum_{n=1}^{\infty} T_n'(t)\sin(n\pi) = 0 & \text{(says nothing; zero=zero)} \end{cases}$$

$$\text{IC:} \quad \sum_{n=1}^{\infty} T_n(0)\sin(n\pi x) = \phi(x).$$

# Step 3 (Find the responce $u_n(x,t) = T_n(t)X_n(x)$)

- From PDE and IC it follows that $T_n(t)$ will satisfy the simple initial value problem

$$T_n' + (n\pi\alpha)^2 T_n = f_n(t)$$

$$T_n(0) = 2\int_0^1 \phi(x)\sin(n\pi x)dx =: a_n$$

- This ODE problem has the solution

$$T_n(t) = a_n e^{-(n\pi\alpha)^2 t} + \int_0^1 e^{-(n\pi\alpha)^2(t-\tau)} f_n(\tau)d\tau.$$

# Step 4 (Find the solution $u(x,t)$)

- Hence, the solution of problem 7-2 is

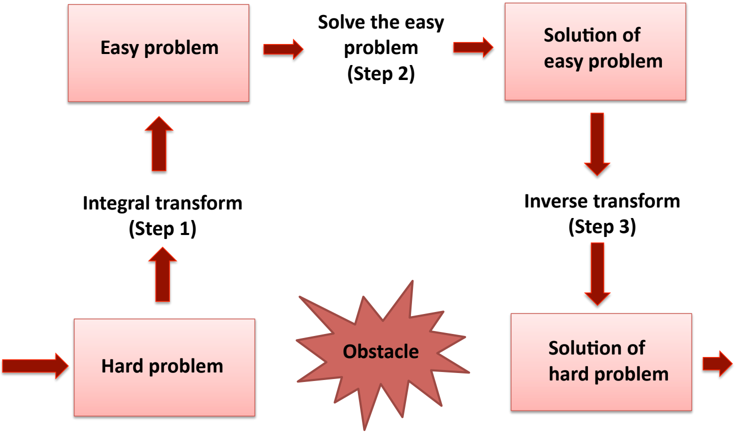
$$u(x,t) = \sum_{n=1}^{\infty} \left[ a_n e^{-(n\pi\alpha)^2 t} \sin(n\pi x) \right] + \sum_{n=1}^{\infty} \left[ \sin(n\pi x) \int\limits_0^1 e^{-(n\pi\alpha)^2(t-\tau)} f_n(\tau) d\tau \right].$$

- Here $\sum_{n=1}^{\infty} \left[ a_n e^{-(n\pi\alpha)^2 t} \sin(n\pi x) \right]$ is the transient part (because of IC), and $\sum_{n=1}^{\infty} \left[ \sin(n\pi x) \int\limits_0^1 e^{-(n\pi\alpha)^2(t-\tau)} f_n(\tau) d\tau \right]$ is the „steady state“ (because of the right-hand side $f(x,t)$).

### Remarks

- The method of eigenfunction expansion is one of the most powerful for solving nonhomogeneous PDEs.
- The eigenfunctions $X_n(x)$ in the expansion change from problem to problem and depend on PDE and BCs.

# General philosophy of transforms



# Integral transformation

- An integral transformation is a transformation that assigns to one function $f(t)$ a new function $F(s)$ by means of a formula like

$$f(t) \to F(s) = \int_A^B K(s,t) f(t) dt$$

  Note that we start with a function of $t$ and end with a function of $s$.

- The function $K(s,t)$ is called the kernel of transformation. It is the major ingredient that distinguishes one transform from another; it is chosen so that the transform has certain properties.

- The limits $A$ and $B$ also change from transformation to transformation.

# Integral transformation

- The general philosophy behind integral transformations is that they eliminate partial derivatives with respect to one of the variables; hence, the new equation has one less variable.

**Example**

If we apply a transform to the PDE

$$u_t = u_{xx}$$

for the purpose of eliminating the time derivative, then we would arrive at an ODE in $x$.

- The transform and its inverse together form what is called a transform pair.

# Sine and Cosine transforms

$$\begin{cases} \mathcal{F}_s[f] = F(\omega) = \dfrac{2}{\pi} \displaystyle\int\limits_0^\infty f(t) \sin(\omega t) dt & \text{(Fourier sine transform)} \\ \mathcal{F}_s^{-1}[F] = f(t) = \displaystyle\int\limits_0^\infty F(\omega) \sin(\omega t) d\omega & \text{(inverse sine transform)} \end{cases}$$

$$\begin{cases} \mathcal{F}_c[f] = F(\omega) = \dfrac{2}{\pi} \displaystyle\int\limits_0^\infty f(t) \cos(\omega t) dt & \text{(Fourier cosine transform)} \\ \mathcal{F}_c^{-1}[F] = f(t) = \displaystyle\int\limits_0^\infty F(\omega) \cos(\omega t) d\omega & \text{(inverse cosine transform)} \end{cases}$$

# Sine and Cosine transforms of derivatives

1. $\mathcal{F}_s[f'] = -\omega \mathcal{F}_c[f]$ (proved by integration by parts)
2. $\mathcal{F}_s[f''] = \frac{2}{\pi}\omega f(0) - \omega^2 \mathcal{F}_s[f]$
3. $\mathcal{F}_c[f'] = -\frac{2}{\pi} f(0) + \omega \mathcal{F}_s[f]$
4. $\mathcal{F}_c[f''] = -\frac{2}{\pi}\omega f(0) - \omega^2 \mathcal{F}_c[f]$

# Fourier Sine Transform

| | $f(x) = \int_0^\infty F(\omega) \sin(\omega x) dx$ <br> $0 < x < \infty$ | $F(\omega) = \frac{2}{\pi} \int_0^\infty f(x) \sin(\omega x) d\omega$ <br> $0 < \omega < \infty$ |
|---|---|---|
| 1. | $f(ax)$ | $\frac{1}{a} F\left(\frac{\omega}{a}\right)$ |
| 2. | $e^{-ax}$ | $\frac{2\omega}{\pi(a^2 + \omega^2)}$ |
| 3. | $x^{-1/2}$ | $\sqrt{\frac{2}{\pi\omega}}$ |
| 4. | $H(a - x)$ | $\frac{2}{\pi\omega} [1 - \cos(\omega a)]$ |
| 5. | $x^{-1}$ | 1 |

# Fourier Sine Transform (cont.)

| | $f(x) = \int\limits_0^\infty F(\omega)\sin(\omega x)dx$ <br> $0 < x < \infty$ | $F(\omega) = \frac{2}{\pi}\int\limits_0^\infty f(x)\sin(\omega x)d\omega$ <br> $0 < \omega < \infty$ |
|---|---|---|
| 6. | $\dfrac{x}{x^2 + a^2}$ | $e^{-a\omega}$ |
| 7. | $\dfrac{x}{x^4 + 4}$ | $\frac{1}{2}e^{-\omega}\sin(\omega)$ |
| 8. | $\tan^{-1}\left(\frac{a}{x}\right)$ | $\dfrac{1 - e^{-a\omega}}{\omega}$ |
| 9. | $-x^2 f(x)$ | $\dfrac{2}{\pi}F''(\omega)$ |

# Fourier Sine Transform (cont.)

| | $f(x) = \int_0^\infty F(\omega)\sin(\omega x)dx$ <br> $0 < x < \infty$ | $F(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\sin(\omega x)d\omega$ <br> $0 < \omega < \infty$ |
|---|---|---|
| 10. | $\operatorname{erfc}\left(\frac{x}{2\sqrt{a}}\right)$ | $\frac{2}{\pi}\left[\frac{1 - e^{-a\omega^2}}{\omega}\right], \quad a > 0$ |

Here

$$H(a - x) = \begin{cases} 1, & x \leqslant a \\ 0, & x > a \end{cases} \quad \text{(Reflected Heaviside function)},$$

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}dt \quad \text{(complimentary-error function)}.$$

# Fourier Cosine Transform

| | $f(x) = \int_0^\infty F(\omega)\cos(\omega x)dx$ <br> $0 < x < \infty$ | $F(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\cos(\omega x)d\omega$ <br> $0 < \omega < \infty$ |
|---|---|---|
| 1. | $f(ax)$ | $\frac{1}{a}F\left(\frac{\omega}{a}\right)$ |
| 2. | $e^{-ax}$ | $\dfrac{2a}{\pi(a^2+\omega^2)}$ |
| 3. | $x^{-1/2}$ | $\sqrt{\dfrac{2}{\pi\omega}}$ |
| 4. | $H(a-x)$ | $\dfrac{2}{\pi\omega}\sin(a\omega)$ |
| 5. | $\delta(x)$ | $2/\pi$ |

# Fourier Cosine Transform (cont.)

| | $f(x) = \int\limits_0^\infty F(\omega)\cos(\omega x)dx$ <br> $0 < x < \infty$ | $F(\omega) = \frac{2}{\pi}\int\limits_0^\infty f(x)\cos(\omega x)d\omega$ <br> $0 < \omega < \infty$ |
|---|---|---|
| 6. | $e^{-ax^2}$ | $\frac{1}{\sqrt{\pi a}}e^{-\omega^2/(4a)}$ |
| 7. | $\frac{\sin(ax)}{x}$ | $H(a-\omega)$ |
| 8. | $\frac{a}{x^2+a^2}$ | $e^{-a\omega}$ |
| 9. | $-x^2 f(x)$ | $\frac{2}{\pi}F''(\omega)$ |

# Solution of an Infinite-Diffusion Problem via the Sine Transform

We now show how the sine transform can solve an important IBVP (the infinite diffusion problem).

**Problem 7-3**

To find the function $u(x, t)$ that satisfies

$$\text{PDE:} \quad u_t = \alpha^2 u_{xx}, \qquad 0 < x < \infty, \quad 0 < t < \infty$$

$$\text{BC:} \quad u(0, t) = A, \qquad 0 < t < \infty$$

$$\text{IC:} \quad u(x, 0) = 0, \qquad 0 \leqslant x \leqslant \infty$$

To solve this, we carry out the following steps.

# Step 1. (Transformation)

- We transform the $x$-variable via the Fourier sine transform so that we get an ODE in time.

$$\mathcal{F}_s[u] = \frac{2}{\pi}\int\limits_0^{\infty} u(x,t)\sin(\omega, x)dx =: U(\omega, t) = U(t),$$

$$\mathcal{F}_s[u_t] = \frac{2}{\pi}\int\limits_0^{\infty} u_t(x,t)\sin(\omega, x)dx$$

$$= \frac{\partial}{\partial t}\left[\frac{2}{\pi}\int\limits_0^{\infty} u(x,t)\sin(\omega, x)dx\right] = \frac{d}{dt}\mathcal{F}_s[u] = \frac{d}{dt}U(t),$$

$$\mathcal{F}_s[u_{xx}] = \frac{2}{\pi}\omega u(0,t) - \omega^2\mathcal{F}_s[u] = \frac{2A\omega}{\pi} - \omega^2 U(t).$$

# Step 1. (Transformation)

- Transformation of the IC provides

$$\mathcal{F}_s[u(x,0)] = U(0) = 0.$$

- Substituting all these expressions into our IBVP, we change the original problem into an initial-value problem

$$\begin{aligned} \text{ODE:} \quad & U'(t) = \alpha^2 \left[ -\omega^2 U(t) + \frac{2A\omega}{\pi} \right] \\ \text{IC:} \quad & U(0) = 0. \end{aligned} \tag{7.2}$$

# Step 2. (Solving the IVP for ODE)

- To solve (7.2), we could use a variety of elementary techniques from ODEs (integrating factor, homogeneous and particular solution); in any case, the solution is

$$U(t) = \frac{2A}{\pi\omega}\left(1 - e^{-\omega^2\alpha^2 t}\right).$$

## Step 3. (Inverse transform)

- The last step is to find the inverse transform of $U(t)$; that is,

$$u(x,t) = \mathcal{F}_s^{-1}[U].$$

We can either evaluate the inverse transform directly from the integral or else resort to the tables. Using the tables we see that

$$\boxed{u(x,t) = A\,\text{erfc}\left(\frac{x}{2\alpha\sqrt{t}}\right)}.$$

## Remark

- The exact values of the complimentary-error function

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2 dt}$$

  can be found in special tables.
- There is also the so-called error function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2 dt}$$

- $\operatorname{erf}(x) + \operatorname{erfc}(x) = 1$.
- The integrals in $\operatorname{erf}(x)$ and $\operatorname{erfc}(x)$ cannot be integrated by the usual elementary tricks of calculus.